

$\mathcal{N} = 4$ super-Yang-Mills in LHC superspace

Part I: Classical and quantum theory*

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Abstract

We present a formulation of the maximally supersymmetric $\mathcal{N} = 4$ gauge theory in Lorentz harmonic chiral (LHC) superspace. It is closely related to the twistor formulation of the theory but employs the simpler notion of Lorentz harmonic variables. They parametrize a two-sphere and allow us to handle efficiently infinite towers of higher-spin auxiliary fields defined on ordinary space-time. In this approach the chiral half of $\mathcal{N} = 4$ supersymmetry is manifest. The other half is realized non-linearly and the algebra closes on shell. We give a straightforward derivation of the Feynman rules in coordinate space. We show that the LHC formulation of the $\mathcal{N} = 4$ super-Yang-Mills theory is remarkably similar to the harmonic superspace formulation of the $\mathcal{N} = 2$ gauge and hypermultiplet matter theories. In the twin paper [1] we apply the LHC formalism to the study of the non-chiral multipoint correlation functions of the $\mathcal{N} = 4$ stress-tensor supermultiplet.

*Dedicated to the memory of Boris Zupnik, a colleague and a friend

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1 Introduction

A very old problem in supersymmetric field theory is how to formulate the maximally supersymmetric $\mathcal{N} = 4$ super-Yang-Mills (SYM) theory off shell with full manifest supersymmetry. A counting argument [2] claims that there exist no *finite* sets of fermionic and bosonic auxiliary fields which could remove the mismatch of the off-shell degrees of freedom of the physical fields. The same counting argument [3] in the case of the $\mathcal{N} = 2$ hypermultiplet has been successfully circumvented by introducing *infinite* sets of auxiliary fields, in the framework of harmonic superspace [4].¹ A similar approach exists to $\mathcal{N} = 3$ SYM [10], a theory with the same physical degrees of freedom as $\mathcal{N} = 4$ SYM but with less manifest supersymmetry. However, the $\mathcal{N} = 4$ barrier still remains a formidable challenge.

Here we do not propose a solution to this old problem but perhaps a step towards it. We present a formulation of $\mathcal{N} = 4$ SYM in Lorentz harmonic chiral (LHC) superspace. In it the chiral half of supersymmetry is manifest and realized off shell. The other half is realized non-linearly and the algebra closes modulo equations of motion and gauge transformation. An important advantage of this ‘semi-off-shell’ formulation is that we are able to maintain the full R-symmetry $SU(4)$ combined with manifest chiral supersymmetry, unlike the alternative formulations of $\mathcal{N} = 4$ SYM in terms of $\mathcal{N} = 1, 2, 3$ superfields.

One of our aims is the simple and clear derivation of the Feynman rules which preserve the symmetries. We apply these rules in the twin paper [1] to construct the non-chiral sector of the Born-level correlation functions of the $\mathcal{N} = 4$ stress-tensor multiplet. Our main result is surprisingly simple: the full non-chiral correlator is obtained from its chiral truncation studied in [11] by a Grassmann shift of the space-time coordinates.

We use Lorentz harmonics (LHs) as a convenient tool for introducing infinite sets of auxiliary fields and pure gauges. Our formulation closely follows, in many aspects, that of $\mathcal{N} = 2$ SYM and hypermultiplet matter in harmonic superspace [4, 12]. The main difference is that here the auxiliary and pure gauge fields have arbitrarily high Lorentz spins instead of the $SU(2)$ isospins in the harmonic formulation of $\mathcal{N} = 2$ SYM and matter.

The Lorentz harmonics are auxiliary variables, in addition to the Euclidean space-time R^4 with coordinates $x^{\dot{\alpha}\alpha} = (\sigma_\mu)^{\dot{\alpha}\alpha} x^\mu$ and Lorentz group $SO(4) \sim SU(2)_L \times SU(2)_R$. The harmonics describe the two-sphere $S^2 \sim SU(2)_L/U(1)$. They are defined as matrices $u_\alpha^\pm \in SU(2)_L$ carrying an $SU(2)_L$ index $\alpha = 1, 2$ and a $U(1)$ charge \pm . The harmonic fields $f(x, u)$ are defined on $R^4 \times S^2$ by their harmonic expansion as functions on $SU(2)_L$ homogeneous under the action of the subgroup $U(1)$. For example, a harmonic field of charge +1 has the infinite expansion

$$f^+(x, u) = f^\alpha(x) u_\alpha^+ + f^{(\alpha\beta\gamma)}(x) u_\alpha^+ u_\beta^+ u_\gamma^- + \dots$$

The terms in this expansion carry $SU(2)_L$ spins $1/2, 3/2, \dots$. Thus, a harmonic field is a collection of infinitely many higher-spin fields $f^{\alpha\beta\cdots}(x)$, most of which are pure gauges or auxiliary fields. Fixing the appropriate gauge and eliminating the auxiliary fields via their kinematical field equations, we obtain a description of the theory in terms of a finite

¹For a review see [5]. See also [6, 7, 8, 9] for alternative approaches.

number of ordinary propagating fields. As an example, the correlators of the $\mathcal{N} = 4$ stress-tensor multiplet do not depend on the LHs (they are integrated out). To put it differently, at the end of the calculation we are able to eliminate the infinite sets of auxiliary and pure gauge fields. This clearly shows the auxiliary character of the LHs.

The idea to use LH variables has a long history. Soon after the development of the $\mathcal{N} = 2$ harmonic superspace method, the close connection between harmonics and twistors was observed in [13]. Harmonic variables on S^2 were used to solve the self-dual Yang-Mills equations, as an alternative to the well-known Ward construction of instantons [14]. This approach to self-dual (super)-Yang-Mills was further developed in [15, 16, 17]. Some aspects of the correspondence harmonics/twistors were discussed in [18].

Later on, LHs were used by one of the authors of the present paper to reformulate Siegel's self-dual $\mathcal{N} = 4$ SYM theory [19] in chiral superspace [20]. The theory is described by two dynamical superfields $A_\alpha^+(x, \theta^+, u)$ and $A^{++}(x, \theta^+, u)$. Here $\theta^{+A} = u_\alpha^+ \theta^{\alpha A}$ is the LH projection of the chiral odd variable $\theta^{\alpha A}$ ($A = 1, \dots, 4$ is an $SU(4)$ R-symmetry index). These superfields play the role of gauge connections for the harmonic-projected space-time derivative $\partial_\alpha^+ = u^{+\alpha} \partial / \partial x^{\dot{\alpha}\alpha}$ and the harmonic derivative $\partial^{++} = u^{+\alpha} \partial / \partial u^{-\alpha}$, respectively. The vanishing of the corresponding curvatures is equivalent to the field equations of self-dual $\mathcal{N} = 4$ SYM. These equations are derived from a superspace *Chern-Simons* action.

Our formulation is closely related to the twistor one developed in [21, 22, 23, 24, 25, 26]. The main conceptual difference is how one treats the harmonic variables and their cousins twistors. In the twistor approach one introduces the twistor projective variable $Z^M = (\lambda_\alpha, \mu^{\dot{\alpha}})$ to describe the space CP^3 . Contact with (complexified) space-time is made via the incidence relation $\mu^{\dot{\alpha}} = x^{\dot{\alpha}\alpha} \lambda_\alpha$. The ordinary fields $f(x)$ are obtained from the twistor fields $F(Z)$ by an integral (Penrose) transform. Thus, the concept of space-time reappears at the end, after performing this very non-trivial transform. In contrast, in the LH approach space-time is present from the very beginning, and the integral Penrose transform is replaced by the simpler spherical harmonic expansion on S^2 . In the twistor approach the accent is put on the conformal symmetry (the twistor Z^M is a linear representation of the conformal group $GL(4)$). In the harmonic approach one insists on the manifest Lorentz symmetry of the ordinary fields $f^{\alpha\beta\cdots}(x)$, viewed as harmonic modes on S^2 .

The development of the twistor approach to $\mathcal{N} = 4$ SYM started with the work of Witten [21]. He found the twistor analog of the Chern-Simons action of [20]. Siegel's self-dual action contains a Lagrange multiplier $G^{\alpha\beta}$ for the self-dual YM tensor $F_{\alpha\beta}$ in the form $\int d^4x G^{\alpha\beta} F_{\alpha\beta}$. Witten proposed to add another superspace gauge invariant to the Chern-Simons action, which contains the square of the Lagrange multiplier, $\int d^4x G^{\alpha\beta} G_{\alpha\beta}$. The sum of the two terms is equivalent to the standard YM action up to a topological term, $\int d^4x F^{\alpha\beta} F_{\alpha\beta}$. In a more explicit form the twistor version of the second action term was given by Mason [22]. The detailed supersymmetric construction was elaborated in [23].

If the action term $\int d^4x G^{\alpha\beta} F_{\alpha\beta}$ can be upgraded to a supersymmetric Chern-Simons term in LHC [20] or twistor [21] superspace, the generalization of the term $\int d^4x G^{\alpha\beta} G_{\alpha\beta}$ takes a completely different form. Its Lagrangian density is given by $\log \det (\bar{\partial}^{-1}(\bar{\partial} + \mathcal{A}))$, where $\bar{\partial}$ is a holomorphic derivative on twistor space and \mathcal{A} is its (super)connection. At close look (especially at the most explicit form shown in [23]) we realize that this peculiar

action term has been known for almost 30 years, in the context of $\mathcal{N} = 2$ SYM formulated in harmonic superspace. Switching from twistors to LHs, we can identify the operator $\bar{\partial}$ with the harmonic derivative ∂^{++} and \mathcal{A} with the gauge connection $A^{++}(x, \theta^+, u)$ mentioned above. Then the Lagrangian term $\log \det (\bar{\partial}^{-1}(\bar{\partial} + \mathcal{A}))$ becomes practically identical with the $\mathcal{N} = 2$ SYM harmonic superspace Lagrangian proposed by Zupnik in [12].

This remarkable similarity between the LHC formulation of $\mathcal{N} = 4$ SYM and the $SU(2)$ harmonic formulation of the $\mathcal{N} = 2$ supersymmetric theories goes even farther. In Sect. 6 we show that the $\mathcal{N} = 4$ Chern-Simons term has its analog as the $\mathcal{N} = 2$ hypermultiplet action in harmonic superspace. The common point between the two theories is the use of harmonics on S^2 , either as LHs or as R-symmetry harmonics. We also propose an eight-dimensional SYM theory whose dimensional reduction gives rise either to the LHC formulation of this paper or to the non-chiral $\mathcal{N} = 2$ formulation of $\mathcal{N} = 4$ SYM.

We present the LHC formulation of $\mathcal{N} = 4$ SYM in Sects. 2 and 3. After recalling some basic definitions and properties of Euclidean LHC superspace, we introduce the two dynamical superfields $A_{\alpha}^{+}(x, \theta^{+}, u)$ and $A^{++}(x, \theta^{+}, u)$ as gauge connections. We then construct the supercurvatures and try to match them with the field content of $\mathcal{N} = 4$ SYM. In this way we derive the two defining supercurvature constraints of the theory. As usual in $\mathcal{N} = 4$ SYM, these are also equations of motion. The next step is to find an action from which the constraints follow. As mentioned above, the action consists of the Chern-Simons term first proposed in [20] and of Zupnik's log det term adapted to the LHC setup.

An important aspect of the LHC/twistor formulation of $\mathcal{N} = 4$ SYM is that only the chiral half (Q) of the $\mathcal{N} = 4$ supersymmetry is manifest. Indeed, the dynamical superfields involve only chiral odd variables. As a consequence, the anti-chiral half (\bar{Q}) remains on shell. This phenomenon was already observed in [20] where the \bar{Q} transformation rules for A_{α}^{+} and A^{++} were found. The addition of the log det term modifies the \bar{Q} transformations and the way their algebra closes. We present a detailed discussion of the supersymmetry of the LHC $\mathcal{N} = 4$ SYM action in Sect. 4. We use these results in the twin paper [1] for constructing the full (non-chiral) supermultiplet of the $\mathcal{N} = 4$ stress-energy tensor.

The quantization of the theory is discussed in Sect. 5. We adopt the light-cone gauge (also called CSW or axial gauge) first proposed in [27] and also used in [23]. The gauge-fixing parameter is an anti-chiral commuting spinor $\lambda^{\dot{\alpha}}$. This gauge has the nice feature that all the propagators are just delta functions in x, u, θ space. This is very helpful in application to the Born-level correlators studied in [1] (and earlier in [11]). As shown there, the integrals at the vertices in the Feynman graphs are lifted and we obtain directly the expected rational expressions for the Born-level correlators. We give some simple examples of LHC supergraphs, postponing the main application of the Feynman rules to Ref. [1].

Appendix A summarizes some properties of the harmonic/space-time distributions that we need for the propagators. Appendix B gives a short dictionary between the twistor and harmonic languages, for the sake of the reader with a twistor background.

2 Euclidean Lorentz harmonic chiral superspace

The Euclidean four-dimensional space can be parametrized by coordinates $x^{\dot{\alpha}\alpha}$, where α and $\dot{\alpha}$ are spinor indices of $SU(2)_L$ and $SU(2)_R$ (the Lorentz group is $SO(4) \sim SU(2)_L \times SU(2)_R$). The \mathcal{N} -extended superspace has coordinates

$$x^{\dot{\alpha}\alpha}, \theta^{\alpha A}, \bar{\theta}_A^{\dot{\alpha}}, \quad (2.1)$$

where $A = 1, \dots, \mathcal{N}$ are co- or contravariant indices of the automorphism group $U(\mathcal{N})$ (or $SU(4)$ in the case $\mathcal{N} = 4$). In it one can realize \mathcal{N} -extended supersymmetry in the following way²

$$\delta x^{\dot{\alpha}\alpha} = -\frac{1}{2}(\bar{\epsilon}_A^{\dot{\alpha}} \theta^{\alpha A} + \epsilon^{\alpha A} \bar{\theta}_A^{\dot{\alpha}}), \quad \delta \bar{\theta}_A^{\dot{\alpha}} = \bar{\epsilon}_A^{\dot{\alpha}}, \quad \delta \theta^{\alpha A} = \epsilon^{\alpha A}. \quad (2.2)$$

The corresponding algebra of supercovariant derivatives is

$$\{D_{\alpha A}, D_{\beta B}\} = 0, \quad \{\bar{D}_{\dot{\alpha}}^A, \bar{D}_{\dot{\beta}}^B\} = 0, \quad \{\bar{D}_{\dot{\alpha}}^A, D_{\beta B}\} = \delta_B^A \partial_{\beta \dot{\alpha}}, \quad (2.3)$$

where

$$D_{\alpha A} = \partial_{\alpha A} + \frac{1}{2} \bar{\theta}_A^{\dot{\alpha}} \partial_{\alpha \dot{\alpha}}, \quad \bar{D}_{\dot{\alpha}}^A = \bar{\partial}_{\dot{\alpha}}^A + \frac{1}{2} \theta^{\alpha A} \partial_{\alpha \dot{\alpha}} \quad (2.4)$$

and $\partial_{\alpha A} = \partial / \partial \theta^{\alpha A}$, $\bar{\partial}_{\dot{\alpha}}^A = \partial / \partial \bar{\theta}_A^{\dot{\alpha}}$, $\partial_{\alpha \dot{\alpha}} = \partial / \partial x^{\dot{\alpha}\alpha}$.

We choose to ‘harmonize’ half of the Lorentz group, e.g., the factor $SU(2)_L$.³ We introduce harmonic variables $u^{\pm\alpha}$ defined as two $SU(2)_L$ spinors forming an $SU(2)_L$ matrix:

$$u^{\pm\alpha} \in SU(2)_L : \quad \begin{cases} u^{+\alpha} u_{\alpha}^{-} \equiv u^{+\alpha} \epsilon_{\alpha\beta} u^{-\beta} = 1 \\ (u^{+\alpha})^* = u_{\alpha}^{-}, \quad (u_{\alpha}^{+})^* = -u^{-\alpha} \end{cases} \quad (2.5)$$

(the $SU(2)_L$ spinor indices are raised and lowered with the Levi-Civita tensor, $\epsilon_{12} = -\epsilon^{12} = 1$). The index \pm refers to the charge of these variables with respect to $U(1)_L \subset SU(2)_L$. Thus, the harmonic variables defined in this way describe the compact coset $S^2 \sim SU(2)_L / U(1)_L$. We are going to apply to them the rules of harmonic calculus [4]. Here we give a short summary (see Appendix A for more detail).

Harmonic functions are defined by their harmonic expansion on S^2

$$f^{(q)}(u) = \sum_{n=0}^{\infty} f^{\alpha_1 \dots \alpha_{2n+q}} u_{(\alpha_1}^+ \dots u_{\alpha_{n+q}}^+ u_{\alpha_{n+q+1}}^- \dots u_{\alpha_{2n+q}}^-). \quad (2.6)$$

By definition, they are homogeneous under the action of $U(1)_L$, i.e., they carry a certain charge q (in (2.6) $q \geq 0$). From (2.6) it is clear that the harmonic functions are collections of infinitely many finite-dimensional irreducible representations of $SU(2)_L$ (multispinors).

²To simplify the formulae we skip the imaginary unit in the supersymmetry variation. Consequently, our translation generator $P = \partial_x$ is anti-hermitian. If needed, the imaginary unit can easily be reinserted.

³LHs of this type were first used in [15, 16, 17] for solving the self-dual SYM constraints.

The differential operators compatible with the defining constraint (2.5) are the covariant harmonic derivatives

$$\begin{aligned}\partial^{++} &= u^{+\alpha} \frac{\partial}{\partial u^{-\alpha}} : \quad \partial^{++} u^{+\alpha} = 0, \quad \partial^{++} u^{-\alpha} = u^{+\alpha} \\ \partial^{--} &= u^{-\alpha} \frac{\partial}{\partial u^{+\alpha}} : \quad \partial^{--} u^{+\alpha} = u^{-\alpha}, \quad \partial^{--} u^{-\alpha} = 0.\end{aligned}\tag{2.7}$$

Together with the charge operator ∂^0 ($\partial^0 u^{\pm\alpha} = \pm u^{\pm\alpha}$) the harmonic derivatives form the algebra of $SU(2)$ realized on the indices \pm of the harmonics,

$$[\partial^{++}, \partial^{--}] = \partial^0.\tag{2.8}$$

The derivatives ∂^{++} and ∂^{--} have the meaning of the raising and lowering operators of $SU(2)_L$, while ∂^0 counts the $U(1)_L$ charge of the harmonic functions, $\partial^0 f^{(q)}(u) = q f^{(q)}(u)$. The restriction to functions on $SU(2)$ with definite charge gives a particular realization of the harmonic coset $SU(2)_L/U(1)_L$.

A direct consequence of the above definitions is the following lemma:

$$\partial^{++} f^{(q)}(u) = 0 \quad \Rightarrow \quad \begin{cases} f^{(q)}(u) = 0, & q < 0 \\ f^{(q)}(u) = f^{\alpha_1 \dots \alpha_q} u_{\alpha_1}^+ \dots u_{\alpha_q}^+, & q \geq 0 \end{cases}.\tag{2.9}$$

This condition can be interpreted as defining the highest weight of a finite-dimensional unitary irreducible representation of $SU(2)$ of spin $q/2$. This irrep is carried by the totally symmetric tensor (multispinor) coefficient $f^{\alpha_1 \dots \alpha_q}$.

Finally, harmonic integration amounts to projecting out the singlet part of a chargeless integrand, according to the rule

$$\int du f^{(q)}(u) = \begin{cases} 0, & q \neq 0 \\ f_{\text{singlet}}, & q = 0 \end{cases}.\tag{2.10}$$

This integration rule is designed to give an $SU(2)$ invariant result. It is compatible with integration by parts for the harmonic derivative ∂^{++} ,

$$\int du \partial^{++} f^{(-2)}(u) = 0,\tag{2.11}$$

due to the absence of a singlet term in the harmonic expansion of the function $f^{(-2)}(u)$.

Now, let us come back to the superspace (2.1). The structure of the algebra (2.3) admits realizations of the \mathcal{N} -extended supersymmetry algebra in subspaces of the superspace involving only part of the Grassmann variables θ . One of them is the chiral superspace which does not contain the variables $\bar{\theta}_A^{\dot{\alpha}}$. It is characterized by the coordinate shift

$$\text{Chiral basis: } x_L^{\dot{\alpha}\alpha} = x^{\dot{\alpha}\alpha} - \frac{1}{2} \bar{\theta}_A^{\dot{\alpha}} \theta^{\alpha A}.\tag{2.12}$$

In the chiral superspace the supersymmetry transformations are

$$\delta x_L^{\alpha\dot{\alpha}} = -\bar{\epsilon}_A^{\dot{\alpha}} \theta^{\alpha A}, \quad \delta \theta^{\alpha A} = \epsilon^{\alpha A}, \quad \delta \bar{\theta}_A^{\dot{\alpha}} = \bar{\epsilon}_A^{\dot{\alpha}}. \quad (2.13)$$

In this basis the antichiral covariant derivative becomes $\bar{D}_\alpha^A = \bar{\partial}_\alpha^A$, therefore the *chiral* superfields defined by the constraint

$$\bar{D}_\alpha^A \Phi = 0 \quad \Rightarrow \quad \Phi = \Phi(x_L^{\dot{\alpha}\alpha}, \theta^{\alpha A}, u) \quad (2.14)$$

do not depend on $\bar{\theta}_A^{\dot{\alpha}}$.

The LHs u^\pm allow us to make a step further and define *chiral-analytic* ‘semi-superfields’ which depend only on one quarter of the Grassmann variables,

$$\Phi(x_L^{\dot{\alpha}\alpha}, \theta^{+A}, u), \quad \text{where } \theta^{+A} = u_\alpha^+ \theta^{\alpha A}. \quad (2.15)$$

We call such objects ‘semi-superfields’ because such an extreme shortening of a superfield is not possible without putting it on shell, if we insist on maintaining the full supersymmetry manifest. Indeed, in addition to the chirality constraint (2.14) it should satisfy the *Lorentz analyticity*⁴ shortening condition

$$D_A^+ \Phi = 0, \quad \text{where } D_A^+ = u_\alpha^+ D_A^\alpha = \partial_A^+ + \bar{\theta}_{\dot{\alpha}A} \partial^{+\dot{\alpha}}. \quad (2.16)$$

Here and it what follows we use the harmonic projected spinor and space-time derivatives

$$\partial_A^\pm = u^{\pm\alpha} \frac{\partial}{\partial \theta^{\alpha A}}, \quad \partial_\alpha^\pm = u^{\pm\alpha} \frac{\partial}{\partial x_L^{\dot{\alpha}\alpha}}.$$

The compatibility of these constraints and the algebra (2.3) imply $\{\bar{D}_\alpha^A, D_B^+\} \Phi = -\delta_B^A \partial_\alpha^+ \Phi = 0$, which puts the superfield on shell, since $\square \Phi = 4 \partial^{+\dot{\alpha}} \partial_{\dot{\alpha}}^- \Phi = 0$. What we really mean is that we have set $\bar{\theta} = 0$ by hand, thus dropping the space-time derivative term in (2.16). Of course, this makes the \bar{Q} supersymmetry non-manifest. As mentioned above, this half of supersymmetry will be realized in a non-linear manner on the semi-superfields (2.15) (see Sect. 4). However, our formulation will maintain the Q half manifest. In the absence of $\bar{\theta}$ we cannot say anymore that our superfields are defined in the chiral basis (2.12), so in the sequel we will drop the subscript of x_L .

3 Lorentz harmonic chiral formulation of $\mathcal{N} = 4$ SYM

In this Section we develop the classical $\mathcal{N} = 4$ SYM theory in LHC superspace. We introduce the two dynamical semi-superfields as the gauge connections for the derivatives ∂_α^+ and ∂^{++} . From them we construct the supercurvatures and try to match them with the

⁴The name comes from the analogy with the Grassmann analytic superfields introduced in [4] for the off-shell formulation of theories with $\mathcal{N} = 2$ supersymmetry. There one uses $SU(2)$ harmonics u_A^\pm to project the R symmetry indices of $\theta^{\alpha A}$ and $\bar{\theta}^{\dot{\alpha} A}$. The analytic superfields depend only on $\theta^{\alpha+}, \bar{\theta}^{\dot{\alpha}+}$ (see Sect. 6 for more detail).

physical fields. In this way we derive the two defining constraints of the theory, equivalent to the field equations. We then find a superspace action from which these equations follow. The action consists of a Chern-Simons term for the self-dual part of the equations and a log det (Zupnik) term for the deviation from self-duality. We display the ordinary component field content of the theory in the non-supersymmetric Wess-Zumino gauge.

3.1 Dynamical superfields as gauge super-connections

We start by introducing the main objects of the theory, two chiral-analytic ‘semi-superfields’ depending only on one fourth of the Grassmann variables:

$$A_{\dot{\alpha}}^+ = A_{\dot{\alpha}}^+(x, \theta^+, u), \quad A^{++} = A^{++}(x, \theta^+, u). \quad (3.1)$$

They are defined as the gauge connections for two covariant harmonic-projected derivatives,

$$\nabla_{\dot{\alpha}}^+ = \partial_{\dot{\alpha}}^+ + A_{\dot{\alpha}}^+, \quad \nabla^{++} = \partial^{++} + A^{++} \quad (3.2)$$

with respect to a gauge group with a chiral-analytic parameter

$$\delta A_{\dot{\alpha}}^+ = \partial_{\dot{\alpha}}^+ \Lambda + [A_{\dot{\alpha}}^+, \Lambda], \quad \delta A^{++} = \partial^{++} \Lambda + [A^{++}, \Lambda], \quad \Lambda = \Lambda(x, \theta^+, u). \quad (3.3)$$

The commutators of these gauge covariant derivatives generate super-curvatures:

$$\begin{aligned} [\nabla_{\dot{\alpha}}^+, \nabla^{++}] &= \partial_{\dot{\alpha}}^+ A^{++} - \partial^{++} A_{\dot{\alpha}}^+ + [A_{\dot{\alpha}}^+, A^{++}] =: W_{\dot{\alpha}}^{+3} \\ [\nabla_{\dot{\alpha}}^+, \nabla_{\dot{\beta}}^+] &= \partial_{\dot{\alpha}}^+ A_{\dot{\beta}}^+ - \partial_{\dot{\beta}}^+ A_{\dot{\alpha}}^+ + [A_{\dot{\alpha}}^+, A_{\dot{\beta}}^+] =: \epsilon_{\dot{\alpha}\dot{\beta}} W^{++}. \end{aligned} \quad (3.4)$$

Following the standard logic in supersymmetric gauge theories, we should try to identify the physical components of the $\mathcal{N} = 4$ SYM multiplet with the super-curvatures evaluated at $\theta = 0$. Thus, we immediately see that $W_{\dot{\alpha}}^{+3} = W_{\dot{\alpha}}^{\alpha\beta\gamma} u_{\alpha}^+ u_{\beta}^+ u_{\gamma}^+ + \dots$ is a bosonic curvature of dimension 1 (the dimension of the space-time derivative $\partial_{\alpha\dot{\alpha}}$; the harmonic variables and derivatives are dimensionless), carrying Lorentz spin $(3/2, 1/2)$. Such a field has no match in the $\mathcal{N} = 4$ SYM multiplet, so it should be set to zero.

$$\text{Constraint I:} \quad W_{\dot{\alpha}}^{+3} = \partial_{\dot{\alpha}}^+ A^{++} - \partial^{++} A_{\dot{\alpha}}^+ + [A_{\dot{\alpha}}^+, A^{++}] = 0. \quad (3.5)$$

This is an example of a curvature constraint typical for all supersymmetric gauge theories.

The other super-curvature in (3.4), $W^{++} = F^{\alpha\beta}(x) u_{\alpha}^+ u_{\beta}^+ + \dots$, can be identified with the self-dual half of the Yang-Mills curvature, $F^{\alpha\beta} = \sigma_{\mu\nu}^{\alpha\beta} F^{\mu\nu}$. Indeed, it has the right dimension 2 and Lorentz spin $(1, 0)$, and is made of the gauge superfield $A_{\dot{\alpha}}^+ = \mathcal{A}_{\alpha\dot{\alpha}}(x) u^{\alpha+} + \dots$, whose first component plays the role of the usual gauge field (gluon).

The next question is where to find the six scalars of the $\mathcal{N} = 4$ SYM multiplet. In the standard approach (see, e.g., [28]) to $\mathcal{N} = 4$ SYM they are identified with the super-curvature $W_{AB} = -W_{BA}$, which appears in the anticommutator of two chiral spinor covariant derivatives,

$$\{\nabla_A^{\alpha}, \nabla_B^{\beta}\} = \epsilon^{\alpha\beta} W_{AB}. \quad (3.6)$$

The absence of a curvature symmetric in the Lorentz indices α, β on the right-hand side is the defining constraint of $\mathcal{N} = 4$ SYM in this standard formulation.⁵ In our LHC approach we can construct the curvature W_{AB} directly in terms of the harmonic gauge connection A^{++} . The construction goes through several steps. First of all, we notice that, with the gauge parameter from (3.3) satisfying the analyticity condition $\partial_A^+ \Lambda = 0$, the harmonic projected spinor derivative ∂_A^+ needs no gauge connection. This suggests to project relation (3.6) with the harmonics $u_\alpha^- u_\beta^+$:

$$\{\nabla_A^-, \partial_B^+\} = \partial_B^+ A_A^- = W_{AB}, \quad (3.7)$$

where we have used the defining property (2.5). Notice that the right-hand side in this relation is antisymmetric in AB while the left-hand side is not. This implies the constraint $\partial_{(B}^+ A_{A)}^- = 0$, which we solve explicitly by constructing the gauge super-connection A_A^- . This is done by covariantizing the obvious commutation relation

$$[\partial^{--}, \partial_A^+] = \partial_A^- \Rightarrow [\nabla^{--}, \partial_A^+] = \nabla_A^- \Rightarrow A_A^- = -\partial_A^+ A^{--}. \quad (3.8)$$

In this way we have introduced yet another gauge super-connection, for the second harmonic derivative (the lowering operator of $SU(2)_L$):

$$\nabla^{--} = \partial^{--} + A^{--}, \quad \delta A^{--} = \partial^{--} \Lambda + [A^{--}, \Lambda]. \quad (3.9)$$

To determine it we covariantize the $SU(2)$ algebraic relation (2.8),

$$[\nabla^{++}, \nabla^{--}] = \partial^0. \quad (3.10)$$

Notice that the charge operator ∂^0 needs no gauge connection since the gauge parameter is a harmonic function of charge zero, $\partial^0 \Lambda = 0$. The commutation relation (3.10) implies a harmonic differential equation for the unknown harmonic connection A^{--} in terms of the given A^{++} :

$$\partial^{++} A^{--} - \partial^{--} A^{++} + [A^{++}, A^{--}] = 0. \quad (3.11)$$

Having in mind our definition (2.6) of the harmonic functions as expansions in terms of finite-dimensional irreps of $SU(2)$, we can easily convince ourselves that this equation has a unique solution for A^{--} . Indeed, the differential operator ∂^{++} is invertible on a harmonic function with negative charge due to the first property on the right-hand side of (2.9). The details of the solution are given below in Sect. 3.3. Here we only remark that while the gauge connection $A^{++}(x, \theta^+, u)$ is chiral-analytic by definition, the new one depends on all chiral Grassmann variables, $A^{--}(x, \theta^\alpha, u)$.

To summarize the above procedure, we start with the chiral-analytic super-connection $A^{++}(x, \theta^+, u)$, from it we determine $A^{--}(x, \theta^\alpha, u)$, which we then substitute in (3.8) and

⁵In a space-time with Minkowski signature another constraint is needed, the reality condition $W_{AB} = \frac{1}{2}\epsilon_{ABCD}\bar{W}^{CD}$. This point is discussed in Sect. 3.1.1.

(3.7). The result is the following manifestly antisymmetric expression for the super-curvature

$$W_{AB} = \partial_A^+ \partial_B^+ A^{--}, \quad (3.12)$$

which contains the six scalars of the $\mathcal{N} = 4$ SYM multiplet, $W_{AB} = \phi_{AB}(x) + \dots$. Let us check that this curvature is indeed gauge covariant:

$$\delta W_{AB} = \partial_A^+ \partial_B^+ \delta A^{--} = \partial_A^+ \partial_B^+ (\partial^{--} \Lambda + [A^{--}, \Lambda]) = [W_{AB}, \Lambda] \quad (3.13)$$

because $\partial_A^+ \partial_B^+ \partial^{--} \Lambda = 0$ due to the analyticity $\partial_A^+ \Lambda = 0$ and the (anti)commutation relations $[\partial^{--}, \partial_A^+] = \partial_A^-$ and $\{\partial_A^+, \partial_B^-\} = 0$. Further, it is (covariantly) harmonic independent,

$$\nabla^{++} W_{AB} = \partial_A^+ \partial_B^+ \nabla^{++} A^{--} = \partial_A^+ \partial_B^+ \partial^{--} A^{++} = 0, \quad (3.14)$$

where we have used (3.11) and the analyticity of A^{++} .

From the curvature (3.12) with the gauge transformation (3.13) we can construct other covariant curvatures by applying one or two spinor derivatives ∂_A^+ . In particular,

$$\mathbb{W}^{++} \equiv \frac{1}{4!} \epsilon^{ABCD} \partial_A^+ \partial_B^+ W_{CD} = (\partial_A^+)^4 A^{--} = G^{\alpha\beta}(x) u_\alpha^+ u_\beta^+ + \dots \quad (3.15)$$

The component field $G^{\alpha\beta}$ has the same characteristics (dimension 2, Lorentz spin $(1, 0)$, R-symmetry singlet) as the self-dual part of the gauge curvature $F^{\alpha\beta}$. The latter, as we pointed out earlier, is the lowest component of the super-curvature W^{++} from (3.4). Notice that the new curvature \mathbb{W}^{++} is built entirely from the gauge super-connection A^{++} (via A^{--} and the differential equation (3.11), see Sect. 3.3 for the detail), while the old one W^{++} is made from the connection A_α^+ . We should not let two similar components coexist, otherwise we would double the $\mathcal{N} = 4$ SYM multiplet. So, we impose the identification

$$\text{Constraint II :} \quad W^{++} = \omega \mathbb{W}^{++}, \quad (3.16)$$

or in terms of the gauge connections,

$$\partial^{+\dot{\alpha}} A_\alpha^+ + A^{+\dot{\alpha}} A_\alpha^+ = \omega (\partial^+)^4 A^{--}. \quad (3.17)$$

Notice that this is a non-trivial relation between the connections A_α^+ and A^{++} appearing on the left-hand and right-hand side, respectively.

We have left the proportionality constant ω on the right-hand side of (3.16) arbitrary. In the twistor literature (Ref. [21] and thereafter) this constant is treated as a perturbative parameter (coupling). We prefer to introduce the gauge coupling g in the traditional way by rescaling the gauge connections $A \rightarrow gA$ and then dividing the action by g^2 (see the end of Sect. 3.2). Nevertheless, it is helpful to keep ω as a parameter which measures the ‘deviation from self-duality’. Indeed, if we set $\omega = 0$ we obtain a stronger version of the constraint (3.16),

$$W^{++} = 0, \quad (3.18)$$

which yields the self-dual YM equation $F_{\alpha\beta} = 0$, see [20] for details. The relaxed form of the constraint (3.16) simply identifies the components $F_{\alpha\beta} = \omega G_{\alpha\beta}$, which is part of the first-order (Lagrange multiplier) formulation of the YM equations (see eq. (3.41) below).

The fermions (gluinos) from the $\mathcal{N} = 4$ SYM multiplet can be identified with spinor derivatives of the above curvatures. Hitting (3.12) with another derivative ∂_C^+ we obtain the fermionic curvature

$$W^{+A} = \frac{1}{3}\epsilon^{ABCD}\partial_B^+\partial_C^+\partial_D^+A^{--} = \psi_\alpha^A(x)u^{+\alpha} + \dots$$

It starts with the chiral gluino field $\psi^{\alpha A}$. The anti-chiral gluino can be found in the curvature $[\nabla_\alpha^+, \nabla_A^-] = \bar{\psi}_{\alpha A}(x) + \dots$ (see also the discussion of the components in the Wess-Zumino gauge in Sect. 3.4).

In conclusion, we have given sufficient evidence that the $\mathcal{N} = 4$ SYM multiplet is indeed described by the set of chiral-analytic gauge super-connections (3.1), provided that we impose two constraints on the super-curvatures, eqs. (3.5) and (3.16). Being necessary for the correct identification of the super-curvatures with the components of the $\mathcal{N} = 4$ SYM multiplet, these constraints turn out to also impose the equations of motion for A_α^+ and A^{++} . This will become clear in Sect. 3.2 where we discuss the $\mathcal{N} = 4$ SYM action.

3.1.1 Reality properties

One of the constraints defining $\mathcal{N} = 4$ SYM is the reality condition on the six scalars, $\phi_{AB} = \frac{1}{2}\epsilon_{ABCD}\bar{\phi}^{CD}$. Without restricting the super-curvature in (3.6) appropriately the content of the multiplet will be doubled (complexified). In a superspace with Minkowski signature (1,3) the Grassmann variables θ^α transform under the fundamental representation of $SL(2, C)$ and their conjugates $\bar{\theta}_{\dot{\alpha}}$ under the inequivalent anti-fundamental. So, imposing a reality condition on the super-curvature (3.12) would require turning on $\bar{\theta}$, something we do not wish to do in our chiral formulation. An easy way out is to change the signature to (2,2). There the Lorentz group is $SL(2, R) \times SL(2, R)$ and the spinors θ^α as well as the super-connections can be real. This choice was made in [19] and [20], however, it requires using LHs on the non-compact coset $SL(2, R)/R$. Then the harmonic expansions and integration become problematic and have to be treated formally. Here we prefer to keep the Euclidean signature (4,0), for the sake of having a well-defined harmonic analysis on $S^2 \sim SU(2)/U(1)$. A reality condition on our chiral superfields can be imposed if we restrict the R-symmetry group $SU(4)$ of $\mathcal{N} = 4$ SYM to its subgroup $Sp(4)$. The latter has an invariant symplectic tensor Ω_{AB} and one can impose the pseudo-Majorana condition (see, e.g., [29]) $\bar{\theta}_{\alpha A} = \epsilon_{\alpha\beta}\theta^{\beta B}\Omega_{BA}$. This is the option that we adopt in the present paper.

In practice, as we show in Sect. 3.4, the component field content of our super-connections exactly matches that of the $\mathcal{N} = 4$ multiplet (a gluon, 4 chiral and 4 antichiral gluinos and 6 scalars), with the above reality condition implied.

3.2 Action for $\mathcal{N} = 4$ SYM in Lorentz harmonic chiral superspace

The main idea now is to derive the two constraints (3.5) and (3.16) from an action principle. We start with (3.5). We remark that by construction the super-curvature $W_{\dot{\alpha}}^{+3}(x, \theta^+, u)$ is chiral-analytic, just like the gauge super-connections. It seems logical to use $A_{\dot{\alpha}}^+$ as a Lagrange multiplier for the constraint (3.5) in an action of the type

$$\int d^4x du d^4\theta^+ \text{Tr} (A^{+\dot{\alpha}} W_{\dot{\alpha}}^{+3}) . \quad (3.19)$$

We remark that the integrand carries Lorentz charge +4 which cancels that of the Grassmann measure. This is necessary for the harmonic integral to give a non-vanishing result. However, there is an obvious problem with this action term – it is not gauge invariant. This can be repaired by noticing that the action term (3.19) involves three non-Abelian gauge super-connections, A^{++} and $A_{\dot{\alpha}}^+$. So, we might try a gauge invariant action of the Chern-Simons type⁶

$$S_{\text{CS}} = \int d^4x du d^4\theta^+ L_{\text{CS}}(x, \theta^+, u) , \quad (3.20)$$

where

$$L_{\text{CS}}(x, \theta^+, u) = \text{tr} (A^{++} \partial^{+\dot{\alpha}} A_{\dot{\alpha}}^+ - \frac{1}{2} A^{+\dot{\alpha}} \partial^{++} A_{\dot{\alpha}}^+ + A^{++} A^{+\dot{\alpha}} A_{\dot{\alpha}}^+) . \quad (3.21)$$

This action term is gauge invariant modulo total derivatives, including the harmonic derivative ∂^{++} . The property (2.11) of the harmonic integral makes this possible. Now we have a gauge invariant action, but we have created a new problem: the variation with respect to A^{++} yields the self-dual field equation (3.18). So, we have to add yet another term to the action which will give rise to the right constraint (3.16). This means that the new term should be built from the gauge super-connection A^{++} , it should be gauge invariant and its variation should supply the right-hand side of eq. (3.16). How to construct such an invariant?

The answer to this has been known for almost 30 years, but in a different context. The formulation of $\mathcal{N} = 2$ SYM in $\mathcal{N} = 2$ harmonic superspace first proposed in [4] makes use of harmonics on the R-symmetry group $SU(2)$ (and not on the Lorentz group as we do now). The dynamical superfield of $\mathcal{N} = 2$ SYM is the gauge prepotential V^{++} which is the super-connection for the harmonic derivative ∂^{++} , exactly as in (3.2). The gauge invariant action is non-polynomial in V^{++} . The first few terms in its expansion were found in [30], but the complete expansion, as well as a very compact form of the action were proposed by Zupnik in [12]. In our present context it is straightforward to adapt Zupnik's construction of the invariant. Unlike the action term (3.20), which is given by an integral over the chiral-analytic superspace, the new term is an integral over the full chiral superspace:

$$S_Z = \int d^4x d^8\theta L_Z(x, \theta) . \quad (3.22)$$

⁶The action (3.20) was proposed for the first time in [20] as a superspace formulation of $\mathcal{N} = 4$ self-dual SYM, following the component field version of Siegel [19]. An equivalent twistor reformulation of this Chern-Simons self-dual action appeared more recently in [21] and [23].

This part of the action involves non-linear (interaction) terms in A^{++} .⁷ Its explicit form is

$$L_Z = \omega \operatorname{tr} \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \int du_1 \dots du_n \frac{A^{++}(x, \theta_1^+, u_1) \dots A^{++}(x, \theta_n^+, u_n)}{(u_1^+ u_2^+) \dots (u_n^+ u_1^+)}, \quad (3.23)$$

where $\theta_i^{+A} = \theta^{\alpha A}(u_i)_\alpha^+$ with $i = 1, \dots, n$. This action term is local in (x, θ) space but non-local in the harmonic space (each copy of A^{++} depends on its own harmonic variable). The properties of the action term (3.22) are discussed in detail in Sect. 3.3.

The Lagrangian density (3.23) can be rewritten in a compact form by introducing an integral harmonic operator with the following kernel

$$\left(\frac{1}{\partial^{++}} \nabla^{++} \right)_{ab} (u_1, u_2) = \delta_{ab} \delta(u_1, u_2) + \frac{f_{abc} A_c^{++}(x, \theta_2^+, u_2)}{(u_1^+ u_2^+)},$$

where f_{abc} is the structure constant of the gauge group. Then (3.23) takes the form [12]⁸

$$L_Z = \omega \log \det \left(\frac{1}{\partial^{++}} \nabla^{++} \right). \quad (3.24)$$

The main claim we are making now is that the sum of the two action terms (3.20) and (3.22) is equivalent to the full $\mathcal{N} = 4$ SYM action,

$$S_{\mathcal{N}=4} = S_{\text{CS}} + S_Z. \quad (3.25)$$

In Sect. 3.4 we examine the component field content of the theory and show that the action (3.25) is indeed equivalent to the $\mathcal{N} = 4$ SYM action in a first-order formulation for the gauge field.

3.2.1 Coupling constant

In the above we have not displayed the gauge coupling constant. To see it, we need to rescale both gauge super-connections, $A \rightarrow gA$ and divide the action by g^2 :

$$\begin{aligned} \frac{1}{g^2} S_{\mathcal{N}=4}(gA) = & \operatorname{tr} \int d^4 x du d^4 \theta^+ (A^{++} \partial^{+\dot{\alpha}} A_{\dot{\alpha}}^+ - \frac{1}{2} A^{+\dot{\alpha}} \partial^{++} A_{\dot{\alpha}}^+) \\ & - \frac{\omega}{2} \operatorname{tr} \int d^4 x d^8 \theta du_1 du_2 \frac{A^{++}(x, \theta_1^+, u_1) A^{++}(x, \theta_2^+, u_2)}{(u_1^+ u_2^+)^2} + O(g). \end{aligned} \quad (3.26)$$

This definition associates the gauge coupling g with the non-Abelian color structure. Every color commutator is accompanied by a factor of g . The bilinear terms shown in (3.26) exist

⁷The cubic term in the Chern-Simons action (3.20) is an interaction between A^{++} and $A_{\dot{\alpha}}^+$ which can be eliminated in the light-cone gauge, see Sect. 5.1. Thus, the true interaction is only in (3.22).

⁸The type of gauge invariant action (3.22), (3.24) is close in spirit to Witten's formulation of the Wess-Zumino model [31], although the context is quite different. An on-shell version of the invariant was considered in [32] for the purpose of reproducing the MHV gluon scattering amplitudes.

also in the Abelian (or free, or $g = 0$) theory. The interaction terms include the cubic term from the Chern-Simons Lagrangian (3.21) and all the non-linear terms from (3.23). We can set the gauge coupling constant $g = 0$ and still have both terms in (3.25). This is in contrast with the philosophy of the twistor formulation of [21], [22], [23], where the full SYM action is treated as a ‘perturbation around a self-dual background’, $S_{\mathcal{N}=4} = S_{\text{CS}} + S_Z$, where the Lagrangian L_Z is proportional to ω (recall (3.23)). In other words, there the ‘free theory’ ($\omega = 0$) is self-dual SYM, not the usual Abelian gauge theory. However, when quantizing the theory, following [23] we prefer to treat the bilinear term in the second line in (3.26) as an ‘interaction term’ giving rise to a ‘bivalent vertex’, while the propagators will be determined from the terms in the first line. The advantage of this treatment is the simpler form of the propagators (see Sect. 5 for detail).

This is one of the reasons why we keep the constant ω in (3.23) arbitrary. Another reason is that it helps us keep track of the non-polynomial modification terms in the \bar{Q} supersymmetry transformations (see Sect. 4.2). This will be important in [1] where we use \bar{Q} to reconstruct the full non-chiral stress-tensor multiplet. There we show that the terms proportional to ω do not contribute to the correlation functions at Born level. If needed, ω can be fixed at some conventional (non-zero) value by computing a correlator and comparing the result to another calculation based on standard Feynman rules.

3.3 Gauge invariance of the $\mathcal{N} = 4$ SYM action

As mentioned earlier, the CS action term (3.20) is gauge invariant in the standard way, up to total derivatives. The invariance of the interaction term (3.22) is less obvious. Here we adapt the argument originally given by Zupnik in [12] (its analog can also be found in the twistor literature).

Let us first compute the variation of L_Z with respect to the superfield A^{++} :

$$\begin{aligned} \delta L_Z &= \omega \operatorname{tr} \sum_{n=2}^{\infty} (-1)^n \int du_1 \dots du_n \frac{\delta A^{++}(x, \theta_1^+, u_1) \dots A^{++}(x, \theta_n^+, u_n)}{(u_1^+ u_2^+) \dots (u_n^+ u_1^+)} \\ &= -\omega \operatorname{tr} \int du \delta A^{++}(x, \theta^+, u) A^{--}(x, \theta, u). \end{aligned} \quad (3.27)$$

Here

$$A^{--}(x, \theta, u) = \sum_{n=1}^{\infty} (-1)^n \int du_1 \dots du_n \frac{A^{++}(1) \dots A^{++}(n)}{(u^+ u_1^+) (u_1^+ u_2^+) \dots (u_n^+ u^+)} \quad (3.28)$$

with $A^{++}(k) = A^{++}(x, \theta \cdot u_k^+, u_k)$, is the gauge super-connection for the harmonic derivative ∂^{--} defined in (3.9). Unlike the connection A^{++} , this one is not chiral-analytic but only chiral, i.e. it depends on the full chiral θ^α . Let us check that it satisfies the defining differential equation (3.11).

For $n \geq 2$ the derivative ∂^{++} acting on the n -th term in (3.28) gives (see (A.15))

$$\begin{aligned}
\partial^{++} A_{(n)}^{--} &\equiv (-1)^n \partial^{++} \int du_1 \dots du_n \frac{A^{++}(1) \dots A^{++}(n)}{(u^+ u_1^+) \dots (u_n^+ u^+)} \\
&= (-1)^n \int du_1 \dots du_n A^{++}(1) \dots A^{++}(n) \\
&\quad \times \left[\frac{\delta(u, u_1)}{(u_1^+ u_2^+) \dots (u_n^+ u^+)} - \frac{\delta(u, u_n)}{(u^+ u_1^+) \dots (u_{n-1}^+ u_n^+)} \right] \\
&= -A^{++} A_{(n-1)}^{--} + A_{(n-1)}^{--} A^{++}, \quad n \geq 2.
\end{aligned}$$

For $n = 1$ we differentiate the harmonic distribution using (A.17) and (A.12), afterwards we integrate the harmonic derivative by parts

$$\begin{aligned}
\partial^{++} A_{(1)}^{--} &\equiv \partial^{++} \int du_1 \frac{A^{++}(u_1)}{(u^+ u_1^+)^2} = \int du_1 \partial^{--} \delta^{(2,-2)}(u, u_1) A^{++}(u_1) \\
&= \partial^{--} A^{++}.
\end{aligned}$$

Putting all of this together we see that (3.11) is indeed satisfied.

Now, consider a variation δA^{++} corresponding to the gauge transformation (3.3),

$$\delta A^{++} = \partial^{++} \Lambda + [A^{++}, \Lambda] = \nabla^{++} \Lambda.$$

Under the trace in (3.27) we can integrate the covariant harmonic derivatives by parts and then use (3.11) in the form $\nabla^{++} A^{--} = \partial^{--} A^{++}$ to obtain

$$\delta_\Lambda L_Z = \omega \operatorname{tr} \int du \Lambda \partial^{--} A^{++}. \quad (3.29)$$

This variation does not vanish by itself. To make it vanish we need to act upon $\partial^{--} A^{++}$ with at least two spinor derivatives ∂_A^+ . The mechanism is the same as in (3.13). In the action (3.22) we have four such derivatives. Indeed, we can rewrite the integration measure in the form

$$\int du d^4 x d^8 \theta = \int du d^4 x (\partial_\theta)^8 = \int du d^4 x (\partial^-)^4 (\partial^+)^4.$$

The four derivatives $(\partial^+)^4$ are more than sufficient to kill $\partial^{--} A^{++}$. In summary, the gauge invariance of S_Z is based on the following property of the gauge variation of L_Z :

$$\delta_{\text{gauge}} L_Z = \int du \Delta L, \quad \partial_A^+ \partial_B^+ \Delta L = 0, \quad (3.30)$$

i.e. ΔL is at most a linear function of θ^{-A} .

Finally, let us perform a general variation of the action term (3.22) with respect to the gauge super-connection A^{++} . Using (3.27), we find

$$\begin{aligned}\delta S_Z &= -\omega \operatorname{tr} \int dud^4x (\partial^-)^4 (\partial^+)^4 [\delta A^{++}(x, \theta^+, u) A^{--}(x, \theta, u)] \\ &= -\omega \operatorname{tr} \int dud^4x d^4\theta^+ \delta A^{++} (\partial^+)^4 A^{--},\end{aligned}$$

hence (see (3.12) and (3.15))

$$\frac{\delta S_Z}{\delta A^{++}} = -\omega (\partial^+)^4 A^{--} = -\omega \mathbb{W}^{++}.$$

Recalling that the variation of the Chern-Simons term (3.20) produces the super-curvature W^{++} , we see that the variation of the full action (3.25) yields the desired constraint (3.16). In principle, one can perform a thorough analysis of the equations of motion (3.5) and (3.16) and obtain all the component field equations, but it is simpler to reveal the component content of the action itself.

3.4 Component field content

Let us explore the component content of the super-connections $A_\alpha^+(x, \theta^+, u)$, $A^{++}(x, \theta^+, u)$. We first consider the simplified example of $\mathcal{N} = 1$ supersymmetry. The harmonic connection has a very short Grassmann expansion:

$$A^{++} = a^{++}(x, u) + \theta^+ \sigma^+(x, u). \quad (3.31)$$

The fields in (3.31) are harmonic, i.e., they contain infinitely many ordinary fields (recall (2.6)). However, we still have the gauge transformations (3.3) with parameter

$$\Lambda = \lambda(x, u) + \theta^+ \rho^-(x, u). \quad (3.32)$$

Let us compare the harmonic expansions (2.6) in (3.31) and (3.32) (bosons only):

$$\begin{aligned}a^{++}(x, u) &= u_\alpha^+ u_\beta^+ a^{\alpha\beta}(x) + u_\alpha^+ u_\beta^+ u_\gamma^+ u_\delta^- a^{\alpha\beta\gamma\delta}(x) + \dots, \\ \lambda(x, u) &= \lambda(x) + u_\alpha^+ u_\beta^- \lambda^{\alpha\beta}(x) + u_\alpha^+ u_\beta^+ u_\gamma^- u_\delta^- \lambda^{\alpha\beta\gamma\delta}(x) + \dots.\end{aligned}$$

Clearly, the parameter $\lambda(x, u)$ contains enough components to completely gauge away the harmonic field $a^{++}(x, u)$ (note that the singlet part $\lambda(x)$ in $\lambda(x, u)$ is not used in the process; it remains non-fixed and plays the rôle of the ordinary gauge parameter). Similarly, the parameter $\rho^-(x, u)$ can gauge away the entire field $\sigma^+(x, u)$. Thus, we arrive at the following⁹

$$\mathcal{N} = 1 \text{ Wess-Zumino gauge:} \quad A^{++} = 0. \quad (3.33)$$

⁹We call the gauges (3.33) and (3.36) “Wess-Zumino gauges” by analogy with the standard $\mathcal{N} = 1$ WZ gauge. The latter uses the entire freedom in the chiral superfield gauge parameter $\Lambda(x, \theta)$ (and its

The other $\mathcal{N} = 1$ gauge connection has the expansion

$$A_{\dot{\alpha}}^+ = \mathcal{A}_{\dot{\alpha}}^+(x, u) + \theta^+ \bar{\psi}_{\dot{\alpha}}(x, u). \quad (3.34)$$

The harmonic dependence in it can be eliminated with the help of the constraint (3.5). Substituting (3.33) and (3.34) into (3.5), we obtain $\partial^{++} A_{\dot{\alpha}}^+ = 0$, from where follow the harmonic equations

$$\begin{aligned} \partial^{++} \mathcal{A}_{\dot{\alpha}}^+(x, u) = 0 &\Rightarrow \mathcal{A}_{\dot{\alpha}}^+(x, u) = u^{+\alpha} \mathcal{A}_{\alpha\dot{\alpha}}(x), \\ \partial^{++} \bar{\psi}_{\dot{\alpha}}(x, u) = 0 &\Rightarrow \bar{\psi}_{\dot{\alpha}}(x, u) = \bar{\psi}_{\dot{\alpha}}(x). \end{aligned} \quad (3.35)$$

These component fields can be identified with the gluon $\mathcal{A}_{\alpha\dot{\alpha}}(x)$ and the anti-chiral gluino $\bar{\psi}_{\dot{\alpha}}(x)$. The $\mathcal{N} = 1$ SYM multiplet includes also a chiral gluino $\psi_{\alpha}(x)$, but there is no room for it in our LHC formulation. The reason is the use of chiral gauge connections. In such a superfield the span of Lorentz helicities is limited by the maximal power of the chiral odd variable θ^{+A} with $A = 1, \dots, \mathcal{N}$. Only in the maximally supersymmetric case $\mathcal{N} = 4$ we can accommodate the entire self-conjugate SYM multiplet with helicities from -1 to $+1$. As we show below, half of them live in the gauge connection A^{++} , the other half in $\mathcal{A}_{\dot{\alpha}}^+$.

The maximal case $\mathcal{N} = 4$ follows the same pattern. Comparing the harmonic dependence of the components of A^{++} with that of the gauge parameter Λ , we can fix the

$$\mathcal{N} = 4 \text{ WZ gauge:} \quad A^{++} = (\theta^+)^{2AB} \phi_{AB}(x) + (\theta^+)_A^3 u^{-\alpha} \psi_{\alpha}^A(x) + 3(\theta^+)^4 u^{-\alpha} u^{-\beta} G_{\alpha\beta}(x), \quad (3.36)$$

where we use the shorthand notation

$$(\theta^+)^{2AB} = \frac{1}{2!} \theta^{+A} \theta^{+B}, \quad (\theta^+)_A^3 = \frac{1}{3!} \epsilon_{ABCD} \theta^{+B} \theta^{+C} \theta^{+D}, \quad (\theta^+)^4 = \frac{1}{4!} \epsilon_{ABCD} \theta^{+A} \theta^{+B} \theta^{+C} \theta^{+D}.$$

By this we have exhausted the entire freedom in the gauge super-parameter but for its first component, $\Lambda(x, \theta^+, u) = \lambda(x) + \dots$, which plays the role of the ordinary gauge parameter.

The other gauge connection $A_{\dot{\alpha}}^+$ has the expansion

$$A_{\dot{\alpha}}^+ = \mathcal{A}_{\dot{\alpha}}^+(x, u) + \theta^{+A} \bar{\psi}_{\dot{\alpha}A}(x, u) + (\theta^+)^{2AB} B_{\dot{\alpha}AB}^-(x, u) + (\theta^+)_A^3 \tau_{\dot{\alpha}}^{--A}(x, u) + (\theta^+)^4 C_{\dot{\alpha}}^{-3}(x, u). \quad (3.37)$$

We recall that it appears only in the Chern-Simons action term (3.20), the interaction term (3.22) is made of A^{++} . So, let us first examine the component content of (3.20). Inserting the Wess-Zumino gauge (3.36) for A^{++} and the expansion (3.37) of $A_{\dot{\alpha}}^+$ into it and doing

conjugate) to gauge away as much as possible from the real gauge superpotential $V(x, \theta, \bar{\theta})$. Only the ordinary gauge parameter $\lambda(x)$ is left intact as it serves the usual component gauge field $A_{\mu}(x)$. In the twistor approach of [22], [23] this gauge is called ‘space-time’ because it reveals the space-time field content of the theory, as opposed to the fields on twistor space. In the LH approach the notion of space-time is always present. The harmonic expansion on S^2 gives rise to infinitely many space-time auxiliary and gauge degrees of freedom. The WZ gauges (3.33) and (3.36) eliminate the (infinite sets of) gauge degrees of freedom, leaving us with propagating and auxiliary fields only.

the Grassmann integral, we obtain

$$S_{\text{CS}} = \int d^4x du \operatorname{tr} \left\{ \frac{1}{2} \phi^{AB} \nabla^{+\dot{\alpha}} B_{\dot{\alpha}AB}^- + \psi^{-A} \nabla^{+\dot{\alpha}} \bar{\psi}_{\dot{\alpha}A} + 3G^{--} F^{++} - C^{-3\dot{\alpha}} \partial^{++} \mathcal{A}_{\dot{\alpha}}^+ \right. \\ \left. - \tau^{--\dot{\alpha}A} \partial^{++} \bar{\psi}_{\dot{\alpha}A} - \frac{1}{4} B^{-\dot{\alpha}AB} \partial^{++} B_{\dot{\alpha}AB}^- - \phi^{AB} \bar{\psi}_A^{\dot{\alpha}} \bar{\psi}_{\dot{\alpha}B} \right\}, \quad (3.38)$$

where $\phi^{AB} = 1/2 \epsilon^{ABCD} \phi_{CD}$, $\nabla^{+\dot{\alpha}} = \partial^{+\dot{\alpha}} + [\mathcal{A}^{+\dot{\alpha}}, \cdot]$, $F^{++} = \partial^{+\dot{\alpha}} \mathcal{A}_{\dot{\alpha}}^+ + \mathcal{A}^{+\dot{\alpha}} \mathcal{A}_{\dot{\alpha}}^+$. The fields B , C , τ are auxiliary and their variation gives rise to the harmonic equations

$$\partial^{++} \mathcal{A}_{\dot{\alpha}}^+(x, u) = 0, \quad \partial^{++} \bar{\psi}_{\dot{\alpha}A}(x, u) = 0, \quad \partial^{++} B^{-\dot{\alpha}AB}(x, u) = \nabla^{+\dot{\alpha}} \phi^{AB}(x),$$

which allow us to eliminate the harmonic dependence of $\mathcal{A}_{\dot{\alpha}}^+$ and $\bar{\psi}_{\dot{\alpha}A}$ (see eq. (3.35)) and to express B in terms of ϕ . The variation of (3.38) with respect to the physical fields in $\mathcal{A}_{\dot{\alpha}}^+$ gives further harmonic equations, from which we can express the auxiliary fields τ and C in terms of the physical ones. The result is

$$\mathcal{A}_{\dot{\alpha}}^+(x, u) = u^{+\alpha} \mathcal{A}_{\alpha\dot{\alpha}}(x), \quad \bar{\psi}_{\dot{\alpha}A}(x, u) = \bar{\psi}_{\dot{\alpha}A}(x), \\ B_{\dot{\alpha}AB}^-(x, u) = u^{-\alpha} \nabla_{\alpha\dot{\alpha}} \phi_{AB}(x), \quad \tau_{\dot{\alpha}}^{--A}(x, u) = \frac{1}{2} u^{-\alpha} u^{-\beta} \nabla_{\alpha\dot{\alpha}} \psi_{\beta}^A(x), \\ C_{\dot{\alpha}}^{-3}(x, u) = u^{-\alpha} u^{-\beta} u^{-\gamma} \nabla_{\alpha\dot{\alpha}} G_{\beta\gamma}(x). \quad (3.39)$$

Putting this back in (3.38) and doing the harmonic integral (it just picks the Lorentz singlet part, see (2.10)), we arrive at the component action

$$S_{\text{CS}} = \int d^4x \operatorname{tr} \left\{ -\frac{1}{8} \nabla^{\dot{\alpha}\alpha} \phi^{AB} \nabla_{\alpha\dot{\alpha}} \phi_{AB} + \frac{1}{2} \psi_{\alpha}^A \nabla^{\dot{\alpha}\alpha} \bar{\psi}_{\dot{\alpha}A} + G^{\alpha\beta} F_{\alpha\beta} - \phi^{AB} \bar{\psi}_A^{\dot{\alpha}} \bar{\psi}_{\dot{\alpha}B} \right\}. \quad (3.40)$$

If we restrict ourselves to the Chern-Simons action term (3.20), we recover the action for $\mathcal{N} = 4$ self-dual SYM, first given in [19, 33, 34]. The $\mathcal{N} = 4$ self-dual multiplet contains all the helicities from +1 (described by the self-dual field \mathcal{A}) down to -1 (the field G). The latter serves as a Lagrange multiplier for the self-duality condition $F_{\alpha\beta} = 0$ on $\mathcal{A}_{\alpha\dot{\alpha}}$. Similarly, the spinor fields ψ_{α}^A and $\bar{\psi}_{\dot{\alpha}A}$ form a Lagrangian pair.

However, here we are not interested in the self-dual but in the full $\mathcal{N} = 4$ SYM theory. The key step is to relax the self-duality constraint $F_{\alpha\beta} = 0$ following from (3.40) by adding a quadratic term in the Lagrange multiplier G ,¹⁰

$$\int d^4x G^{\alpha\beta} F_{\alpha\beta} \rightarrow \int d^4x [G^{\alpha\beta} F_{\alpha\beta} - \frac{\omega}{2} G^{\alpha\beta} G_{\alpha\beta}]. \quad (3.41)$$

This is the well-known *first-order form* of the YM action (see, e.g., [37]). Now the variation with respect to G simply identifies $\omega G = F$. Eliminating G from (3.41), we obtain the action $\int d^4x F^{\alpha\beta} F_{\alpha\beta} \sim \int d^4x F^{\mu\nu} F_{\mu\nu}$ up to a topological term.

¹⁰In the twistor literature the action (3.41) is referred to as the Chalmers-Siegel action, often quoting [35], although this paper deals only with the self-dual action of Siegel [19]. This first-order form of the YM action appears in the paper [36], but the authors quote an earlier source [37].

We can now clearly see the harmonic superspace analog of the above transition from self-dual to full SYM. The self-duality equation is contained in the super-curvature constraint $W^{++} = 0$ and its relaxed form is the constraint (3.16). As explained above, the right-hand side of (3.16) follows from the variation of the interaction term (3.22). We should then expect that S_Z contains the quadratic Lagrange multiplier term GG from (3.41). Indeed, this is straightforward to show in the WZ gauge (3.37). The bilinear term in (3.22), (3.23) has the form

$$-\frac{\omega}{2}\text{tr} \int d^4x d^8\theta du_1 du_2 \frac{A^{++}(x, \theta_1^+, u_1) A^{++}(x, \theta_2^+, u_2)}{(u_1^+ u_2^+)^2}. \quad (3.42)$$

The Grassmann integral requires $(\theta)^8$ and this can only come from the last term in (3.36), $\int d^8\theta (\theta_1^+)^4 (\theta_2^+)^4 = (u_1^+ u_2^+)^4$. This cancels the singular harmonic factor in (3.42), after which the harmonic integration picks the singlet term and we obtain the quadratic Lagrange multiplier term $G_{\alpha\beta} G^{\alpha\beta}$ from (3.41). Further, the non-linear part of (3.22), (3.23) is truncated at $n = 4$ due to the nilpotent character of A^{++} in the WZ gauge (3.36). The cubic term $(\theta_1^+)^2 \phi(\theta_2^+)^3 \psi(\theta_3^+)^3 \psi$ supplies the chiral Yukawa interaction $\phi\psi\psi$ (the complement of the anti-chiral Yukawa term in (3.40)). Note that $(\theta_1^+)^2 \phi(\theta_2^+)^2 \phi(\theta_3^+)^4 G$ does not contribute due to the harmonic integration (absence of a Lorentz singlet in the product $\phi\phi G$). Finally, the quartic term $(\theta_1^+)^2 \phi(\theta_2^+)^2 \phi(\theta_3^+)^2 \phi(\theta_4^+)^2 \phi$ supplies the $(\phi)^4$ interaction. So, we get¹¹

$$S_Z = \omega \int d^4x \text{tr} \left(-\frac{1}{2} G_{\alpha\beta} G^{\alpha\beta} - \frac{1}{4} \phi_{AB} \psi^{\alpha A} \psi_{\alpha}^B + \frac{1}{32} [\phi^{AB}, \phi^{CD}] [\phi_{AB}, \phi_{CD}] \right). \quad (3.43)$$

Together, the action terms (3.40) and (3.43) form the complete $\mathcal{N} = 4$ SYM action with a first-order (Lagrange multiplier) formulation of the YM Lagrangian.

3.5 Yang-Mills theory in Lorentz harmonic space

The formulation of $\mathcal{N} = 4$ SYM described above can be truncated to the ordinary (non-supersymmetric) Yang-Mills theory. In doing so we recover the result of [38]. We refer the reader to that paper for the detailed comparison with the original twistor version.

The truncation is done by dropping all the components of the semi-superfields A^{++} and A_{α}^+ which carry $SU(4)$ indices:

$$A^{++} = \mathcal{A}^{++}(x, u) + 3(\theta^+)^4 \mathcal{G}^{--}(x, u), \quad A_{\alpha}^+ = \mathcal{A}_{\alpha}^+(x, u) + (\theta^+)^4 C_{\alpha}^{-3}(x, u). \quad (3.44)$$

Notice that unlike the WZ gauge (3.36), here the connection A^{++} still has a component at level $(\theta)^0$. Together with \mathcal{A}_{α}^+ they play the role of the gauge connections for the covariant harmonic and projected space-time derivatives (cf. (3.2)):

$$\nabla^{++} = \partial^{++} + \mathcal{A}^{++}, \quad \nabla_{\alpha}^+ = \partial_{\alpha}^+ + \mathcal{A}_{\alpha}^+. \quad (3.45)$$

¹¹This coincides with eq. (3.27) in [23]. Likewise, their eq. (3.16) coincides with our (3.40). The latter first appeared in [19, 33, 34] and was recast in superspace form in [20].

They undergo gauge transformation like in (3.3) but with a bosonic parameter $\lambda(x, u)$. The $(\theta^+)^4$ components in (3.44) are Lagrange multipliers, as we show below.

Let us now substitute the truncated expansions (3.44) into the action (3.25) and integrate out the Grassmann variables. The Chern-Simons part (3.20) gives

$$S_{\text{CS}} = \text{tr} \int d^4x du \left[3\mathcal{G}^{--} F^{++} - C^{-3\dot{\alpha}} (\partial^{++} \mathcal{A}_{\dot{\alpha}}^+ - \partial_{\dot{\alpha}}^+ \mathcal{A}^{++} + [\mathcal{A}^{++}, \mathcal{A}_{\dot{\alpha}}^+]) \right] \quad (3.46)$$

with $F^{++} = \frac{1}{2}[\nabla^{+\dot{\alpha}}, \nabla_{\dot{\alpha}}^+] = \partial^{+\dot{\alpha}} A_{\dot{\alpha}}^+ + A^{+\dot{\alpha}} A_{\dot{\alpha}}^+$. The Lagrange multiplier C^{-3} imposes the zero-curvature condition

$$0 = \partial^{++} \mathcal{A}_{\dot{\alpha}}^+ - \partial_{\dot{\alpha}}^+ \mathcal{A}^{++} + [\mathcal{A}^{++}, \mathcal{A}_{\dot{\alpha}}^+] \Leftrightarrow [\nabla^{++}, \nabla_{\dot{\alpha}}^+] = 0. \quad (3.47)$$

This harmonic differential equation for the connection $\mathcal{A}_{\dot{\alpha}}^+$ allows us to express it in terms of the other connection \mathcal{A}^{++} , except for the first term in the harmonic expansion $\mathcal{A}_{\dot{\alpha}}^+(x, u) = u^{+\alpha} \mathcal{A}_{\alpha\dot{\alpha}}(x) + \dots$ (cf. (3.39)). This component is identified with the gluon field. The other Lagrange multiplier \mathcal{G}^{--} , if left alone, would impose the self-duality constraint $F^{++} = 0$. However, \mathcal{G}^{--} appears also in the interaction term (3.22), (3.23):

$$S_Z = 9\omega \text{tr} \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \sum_{k \neq j} \int du_1 \dots du_n (u_k^+ u_j^+)^4 \times \frac{\mathcal{A}^{++}(1) \dots \mathcal{A}^{++}(k-1)}{(u_1^+ u_2^+) \dots (u_{k-1}^+ u_k^+)} \frac{\mathcal{G}^{--}(k)}{(u_k^+ u_{k+1}^+)} \frac{\mathcal{A}^{++}(k+1) \dots \mathcal{A}^{++}(j-1)}{(u_{k+1}^+ u_{k+2}^+) \dots (u_{j-1}^+ u_j^+)} \frac{\mathcal{G}^{--}(j)}{(u_j^+ u_{j+1}^+)} \frac{\mathcal{A}^{++}(j+1) \dots \mathcal{A}^{++}(n)}{(u_{j+1}^+ u_{j+2}^+) \dots (u_n^+ u_1^+)}, \quad (3.48)$$

where the argument of each \mathcal{A}^{++} or \mathcal{G}^{--} refers to the harmonic variable at that point (cf. (3.23)). Eqs. (3.46) and (3.48) coincide with eq. (35) in [38]. As shown there, this YM action in Lorentz harmonic space is equivalent to the twistor formulation of [22], [23], [24].

To recover the standard first-order formulation of YM theory in Euclidean space-time, we can profit from the gauge transformation $\delta \mathcal{A}^{++} = \partial^{++} \lambda + [\mathcal{A}^{++}, \lambda]$ to completely gauge away the harmonic connection, $\mathcal{A}^{++} = 0$. This still leaves the first component in $\lambda(x, u) = \lambda(x) + \dots$ as the usual gauge parameter. In this gauge eq. (3.47) becomes trivial yielding $\mathcal{A}_{\dot{\alpha}}^+(x, u) = u^{+\alpha} \mathcal{A}_{\alpha\dot{\alpha}}(x)$. Substituting this back in (3.46) we get

$$S_{\text{CS}} = \text{tr} \int d^4x G_{\alpha\beta} F^{\alpha\beta} \quad \text{with} \quad G_{\alpha\beta}(x) = 3 \int du u_{\alpha}^+ u_{\beta}^+ \mathcal{G}^{--}(x, u).$$

Here the second relation is an example of the harmonic version of a ‘twistor transform’. Further, all but the first (bilinear) term in (3.48) drop out, thus reducing the ‘interaction’ Lagrangian to the square of the Lagrange multiplier G :

$$S_Z = -\frac{9\omega}{2} \text{tr} \int d^4x du_1 du_2 (u_1^+ u_2^+)^2 \mathcal{G}^{--}(1) \mathcal{G}^{--}(2) = -\frac{\omega}{2} \text{tr} \int d^4x G^{\alpha\beta} G_{\alpha\beta}.$$

4 Supersymmetry of the $\mathcal{N} = 4$ SYM action

In this Section we discuss the realization of $\mathcal{N} = 4$ supersymmetry in the action (3.25). We have already shown that in the WZ gauge this action coincides with the full $\mathcal{N} = 4$ SYM component action, which is invariant under $\mathcal{N} = 4$ supersymmetry. The supersymmetry of the component action is not manifest, of course.

As explained at the beginning of Sect. 3.1, the chiral-analytic gauge super-connections (3.1) do not transform as superfields under the full $\mathcal{N} = 4$ supersymmetry, but only under its chiral (Q) half. The other half (\bar{Q}) is realized non-linearly. In addition, the supersymmetry algebra closes modulo field equations and compensating gauge transformations. This is typical for an on-shell realization of supersymmetry. We repeat that the purpose of this article is not to break the (so far unsurmountable) $\mathcal{N} = 4$ barrier for a theory with full off-shell supersymmetry.

In Sect. 4.1 we recall the supersymmetry transformation in the LHC formulation of the self-dual theory from [20]. Then in Sect. 4.2 we explain how \bar{Q} supersymmetry is modified in the full SYM theory. In Sect. 4.3 we verify that the supersymmetry generators form the super-Poincaré algebra on shell and up to compensating gauge transformations. In Sect. 4.4 we demonstrate that the non-linearly realized \bar{Q} supersymmetry commutes with the gauge transformations. Finally, in Sect. 4.5 we show that in the WZ gauge the LHC supersymmetry transformations take the conventional form of the transformation rules for the physical fields. The material of the present Section constitutes our toolbox in [1] where we reconstruct the $\bar{\theta}$ dependence of the non-chiral stress-tensor multiplet formulated in the LHC superspace.

4.1 Supersymmetry of the Chern-Simons action

The way $\mathcal{N} = 4$ supersymmetry is realized on the gauge connections A^{++} and A_α^+ was discussed for the first time in [20], in the context of the self-dual $\mathcal{N} = 4$ SYM theory. There one imposes the constraint (3.5) and the self-dual version (3.18) of the constraint (3.16). The action is given by the term S_{CS} from (3.20), without the interaction term (3.22). In [20] it was shown that S_{CS} is invariant under the supersymmetry transformations

$$\delta_{\text{CS}} A_\alpha^+ = (\bar{\epsilon}_B^\beta \theta^{+B} \partial_\beta^- - u_\beta^+ \epsilon^{\beta B} \partial_B^-) A_\alpha^+, \quad (4.1)$$

$$\delta_{\text{CS}} A^{++} = (\bar{\epsilon}_B^\beta \theta^{+B} \partial_\beta^- - u_\beta^+ \epsilon^{\beta B} \partial_B^-) A^{++} + (\bar{\epsilon}_B^\beta \theta^{+B}) A_\beta^+. \quad (4.2)$$

These are transformation laws of ‘semi-superfields’. We are not transforming the coordinates like in (2.13) but the superfields themselves. From this point of view the harmonic derivative ∂^{++} does not commute with the translation part of (4.1), (4.2),

$$[\partial^{++}, \bar{\epsilon}_B^\beta \theta^{+B} \partial_\beta^- - u_\beta^+ \epsilon^{\beta B} \partial_B^-] = \bar{\epsilon}_B^\beta \theta^{+B} \partial_\beta^+ - u_\beta^+ \epsilon^{\beta B} \partial_B^+. \quad (4.3)$$

Using (4.1), (4.2), (4.3) it is easy to check that

$$\delta L_{\text{CS}} = (\bar{\epsilon}_B^\beta \theta^{+B} \partial_\beta^- - u_\beta^+ \epsilon^{\beta B} \partial_B^-) L_{\text{CS}} + \frac{1}{2} (\bar{\epsilon}_B^\beta \theta^{+B}) \partial^{+\dot{\alpha}} \text{tr}(A_\alpha^+ A_{\dot{\beta}}^+), \quad (4.4)$$

i.e., the Lagrangian transforms into a total derivative with respect to the variables $x^{\dot{\alpha}\alpha}$ and θ^{+A} . Consequently, the action (3.20) is invariant.

The $\mathcal{N} = 4$ supersymmetry transformations (4.1), (4.2) form a closed algebra on shell (see Sect. 4.3), i.e. using the field equation (3.5) and the self-dual version (3.18) of (3.16):

$$\begin{aligned} [\delta_{\text{CS}}(\kappa), \delta_{\text{CS}}(\epsilon)] A_{\dot{\alpha}}^+ &= (\epsilon^{\beta B} \bar{\kappa}_B^{\dot{\beta}} - \kappa^{\beta B} \bar{\epsilon}_B^{\dot{\beta}}) \partial_{\beta \dot{\beta}} A_{\dot{\alpha}}^+ + \delta_{\Lambda} A_{\dot{\alpha}}^+ \\ [\delta_{\text{CS}}(\kappa), \delta_{\text{CS}}(\epsilon)] A^{++} &= (\epsilon^{\beta B} \bar{\kappa}_B^{\dot{\beta}} - \kappa^{\beta B} \bar{\epsilon}_B^{\dot{\beta}}) \partial_{\beta \dot{\beta}} A^{++} + \delta_{\Lambda} A^{++}, \end{aligned}$$

up to a gauge transformation (3.3) with the field-dependent parameter

$$\Lambda_{Q\bar{Q}} = u_{\dot{\beta}}^-(\kappa^{\beta B} \bar{\epsilon}_B^{\dot{\alpha}} - \epsilon^{\beta B} \bar{\kappa}_B^{\dot{\alpha}}) A_{\dot{\alpha}}^+. \quad (4.5)$$

4.2 Full action and modified \bar{Q} supersymmetry transformation

After completing the CS action with the ‘interaction’ term S_Z (3.22), (3.23), we need to modify the \bar{Q} -half of the supersymmetry transformations.¹² The basic reason is that the field equation (3.18) is modified by the interaction term and becomes (3.16). The non-vanishing right-hand side of (3.16) suggests the modification

$$\delta A^{+\dot{\alpha}} = \delta_{\text{CS}} A^{+\dot{\alpha}} + \delta_Z A^{+\dot{\alpha}}, \quad (4.6)$$

where

$$\delta_Z A^{+\dot{\alpha}} = \omega (\partial^+)^4 [(\bar{\epsilon}_B^{\dot{\alpha}} \theta^{-B}) A^{--}], \quad (4.7)$$

while δA^{++} in (4.2) remains unchanged, $\delta A^{++} = \delta_{\text{CS}} A^{++}$ and $\delta_Z A^{++} = 0$. We remark that the modification terms δ_Z are denoted by δ_{int} in [39]. This notation is somewhat misleading because we can switch off the interaction by setting $g = 0$, but the modification term (4.7) clearly does not vanish. In other words, the bilinear part of the action (3.26) is invariant under \bar{Q} supersymmetry (4.2), (4.6) including the linearized part of (4.7).

Let us show that this modification makes the action (3.25) invariant under the full supersymmetry. Evidently, $\delta_Z S_Z = 0$ since S_Z depends only on the connection A^{++} , however (recall the variation of S_Z (3.27))

$$\delta_{\text{CS}} S_Z = -\omega \text{tr} \int d^4 x d u d^8 \theta \left[(\bar{\epsilon}_B^{\dot{\beta}} \theta^{+B}) A_{\dot{\beta}}^+ + (\bar{\epsilon}_B^{\dot{\beta}} \theta^{+B} \partial_{\dot{\beta}}^- - u_{\dot{\beta}}^+ \epsilon^{\beta B} \partial_B^-) A^{++} \right] A^{--}. \quad (4.8)$$

On the other hand (recall that $\nabla^{++} = \partial^{++} + [A^{++}, \cdot]$),

$$\begin{aligned} \delta_Z S_{\text{CS}} &= \text{tr} \int d^4 x d u d^4 \theta^+ \delta_Z A^{+\dot{\alpha}} (\partial_{\dot{\alpha}}^+ A^{++} - \nabla^{++} A_{\dot{\alpha}}^+) \\ &= \omega \text{tr} \int d^4 x d u d^4 \theta^+ (\partial^+)^4 [(\bar{\epsilon}_B^{\dot{\alpha}} \theta^{-B}) A^{--}] (\partial_{\dot{\alpha}}^+ A^{++} - \nabla^{++} A_{\dot{\alpha}}^+) \\ &= \omega \text{tr} \int d^4 x d u d^8 \theta (\bar{\epsilon}_B^{\dot{\alpha}} \theta^{-B}) (\partial_{\dot{\alpha}}^+ A^{++} - \nabla^{++} A_{\dot{\alpha}}^+) A^{--}. \end{aligned} \quad (4.9)$$

¹²Such a modification was proposed in [39], in a component field formulation of the theory. The modification in the form (4.7) has been known for a long time in the context of the formulation of $\mathcal{N} = 4$ SYM in $\mathcal{N} = 2$ harmonic superspace [5], see Sect. 6 for the detail.

In the second line, when completing the measure $d^4\theta^+(\partial^+)^4 = d^8\theta$ we use the analyticity of $\partial_\alpha^+ A^{++} - \nabla^{++} A_\alpha^+$. In the following we omit the full superspace integration measure. Collecting the terms with $\bar{\epsilon}$ and A_α^+ from (4.8) and (4.9), we get

$$-\omega \operatorname{tr} \int [(\bar{\epsilon}_B^{\dot{\alpha}} \theta^{+B}) A_\alpha^+ + (\bar{\epsilon}_B^{\dot{\alpha}} \theta^{-B}) \nabla^{++} A_\alpha^+] A^{-} = -\omega \operatorname{tr} \int \nabla^{++} [(\bar{\epsilon}_B^{\dot{\alpha}} \theta^{-B}) A_\alpha^+] A^{-}.$$

Then we integrate ∇^{++} by parts and use the defining relation (3.11) in the form $\nabla^{++} A^{-} = \partial^{-} A^{++}$ to obtain

$$\omega \int d^8\theta [(\bar{\epsilon}_B^{\dot{\alpha}} \theta^{-B}) A_\alpha^+] \partial^{-} A^{++} = \omega \int d^4\theta^+ \bar{\epsilon}_B^{\dot{\alpha}} A_\alpha^+ (\partial^+)^4 [\theta^{-B} \partial^{-} A^{++}] = 0. \quad (4.10)$$

Here we have used the analyticity of A_α^+ and A^{++} ; one derivative ∂^+ eliminates θ^- , two others annihilate $\partial^{-} A^{++}$.

Further, the terms with A^{++} from (4.8) and (4.9) combine into

$$\begin{aligned} \delta_{\text{CS}} S_Z + \delta_Z S_{\text{CS}} &= -\omega \operatorname{tr} \int d^4 x d u d^8\theta [(\bar{\epsilon}_B^{\dot{\alpha}} \theta^{+B}) \partial_\alpha^- - (\bar{\epsilon}_B^{\dot{\alpha}} \theta^{-B}) \partial_\alpha^+ - u_\beta^+ \epsilon^{\beta B} \partial_B^-] A^{++} \cdot A^{-} \\ &= -\omega \operatorname{tr} \int d^4 x d u d^8\theta [-\bar{\epsilon}_B^{\dot{\alpha}} \theta^{\alpha B} \partial_{\alpha\dot{\alpha}} + \epsilon^{\beta B} \partial_{\beta B}] A^{++} \cdot A^{-}, \end{aligned} \quad (4.11)$$

where in the second line we have used the analyticity condition $\partial_B^+ A^{++} = 0$ to write the Q supersymmetry transformation (4.2) in the form $\delta A^{++} = (-u_\beta^+ \epsilon^{\beta B} \partial_B^- + u_\beta^- \epsilon^{\beta B} \partial_B^+) A^{++} = \epsilon^{\beta B} \partial_{\beta B} A^{++}$. The result is a \bar{Q} shift of x and a Q shift of θ as in (2.13). Now, recall the detailed expression (3.23) for the Lagrangian in (3.22). In it each A^{++} depends on the same $(x_{\alpha\dot{\alpha}}, \theta^{\alpha A})$, as in the integration measure in (4.11), even though in each $A^{++}(i)$ the odd variable $\theta^{\alpha A}$ appears projected with the harmonic $(u_i)_\alpha^+$. Thus, the variation (4.11) can be interpreted as a *harmonic independent* shift $\delta x = -\bar{\epsilon}\theta$, $\delta\theta = \epsilon$ of all A^{++} in (3.23). This shift is a total derivative, so $\int d^4 x d^8\theta$ annihilates it. This completes the proof that the action (3.25) is invariant under the supersymmetry transformations (4.2) and (4.6).

4.3 Closure of the supersymmetry algebra

Here we show that the modified supersymmetry transformations (4.1), (4.2) and (4.6) form the super-Poincaré algebra on shell, up to field-dependent gauge transformations. The supersymmetry generators and variations are related by

$$\delta(\epsilon) = \delta_Q(\epsilon) + \delta_{\bar{Q}}(\epsilon) = \epsilon^{\beta B} Q_{\beta B} + \bar{\epsilon}_B^{\dot{\beta}} \bar{Q}_{\dot{\beta}}^B.$$

The generators of Q supersymmetry are the same for both analytic connections,

$$Q_B^\beta = \partial_B^\beta \rightarrow Q_B^\beta A(\theta^+) = -u^{+\beta} \partial_B^- A(\theta^+), \quad (4.12)$$

while those of \bar{Q} supersymmetry are different for A^{++} and A_α^+ even in the self-dual theory,

$$\bar{Q}_\beta^B A_\alpha^+ = \theta^{+B} \partial_{\dot{\beta}}^- A_\alpha^+ + (\bar{Q}_Z)_\beta^B A_\alpha^+, \quad (\bar{Q}_Z)_\beta^B A_\alpha^+ = \omega(\partial^+)^4 (\theta^{-B} A^{--}) \epsilon_{\alpha\dot{\beta}} \quad (4.13)$$

$$\bar{Q}_\beta^B A^{++} = \theta^{+B} \left(\partial_{\dot{\beta}}^- A^{++} + A_\beta^+ \right). \quad (4.14)$$

The generator $\bar{Q}_\beta^B A^{++}$ can be rewritten with the help of the kinematical field equation (3.5), $\partial_{\dot{\beta}}^+ A^{++} = \nabla^{++} A_\beta^+$, in a different form which is not modified by the interaction term:

$$\begin{aligned} \bar{Q}_\beta^B A^{++} &= -\theta^{\beta B} \partial_{\beta\dot{\beta}} A^{++} + \theta^{-B} \partial_{\dot{\beta}}^+ A^{++} + \theta^{+B} A_\beta^+ = -\theta^{\beta B} \partial_{\beta\dot{\beta}} A^{++} + \theta^{-B} \nabla^{++} A_\beta^+ + \theta^{+B} A_\beta^+ \\ &= -\theta^{\beta B} \partial_{\beta\dot{\beta}} A^{++} + \nabla^{++} (\theta^{-B} A_\beta^+). \end{aligned} \quad (4.15)$$

From (4.15) and the defining relation (3.11) we can find the \bar{Q} transformation of A^{--}

$$\bar{Q}_\beta^B A^{--} = -\theta^{\beta B} \partial_{\beta\dot{\beta}} A^{--} + \nabla^{--} (\theta^{-B} A_\beta^+) \quad \text{with} \quad \nabla^{--} = \partial^{--} + [A^{--}, \cdot]. \quad (4.16)$$

We remark that the terms $\nabla^{\pm\pm} (\theta^{-B} A_\beta^+)$ in (4.15) and (4.16) are not gauge transformations (3.3) because the ‘parameter’ $\theta^{-B} A_\beta^+$ is not LH-analytic.

The Q transformations are just shifts of the odd coordinates, therefore they obviously commute. For the rest of the supersymmetry algebra involving \bar{Q} , we expect that it closes up to field equations and compensating gauge transformations. Let us first check the closure on A^{++} . Making two infinitesimal transformations with parameters ϵ and $\bar{\kappa}$ and using (4.12) and (4.15), we find the commutator of the \bar{Q} and Q transformations

$$[\delta_{\bar{Q}}(\kappa), \delta_Q(\epsilon)] A^{++} = \epsilon^{\beta B} \bar{\kappa}_B^{\dot{\beta}} \partial_{\beta\dot{\beta}} A^{++} + \nabla^{++} \Lambda_{Q\bar{Q}}. \quad (4.17)$$

The calculation is the same as in the self-dual case. Here the first term on the right-hand side is the expected translation according to $\{\bar{Q}, Q\} = \partial_x$, the second term is a compensating gauge transformation with the parameter (4.5).

Further, using (4.14), (4.13) and (3.11) we find the commutator of two \bar{Q} transformations on the connection A^{++} ,

$$[\delta_{\bar{Q}}(\kappa), \delta_{\bar{Q}}(\epsilon)] A^{++} = 2\omega(\partial^+)^4 (\bar{\epsilon}\bar{\kappa})_{BC} \theta^{+B} \theta^{-C} A^{--}, \quad (4.18)$$

with $(\bar{\epsilon}\bar{\kappa})_{BC} = \bar{\epsilon}_{\dot{\beta}B} \bar{\kappa}_C^{\dot{\beta}}$. Then we use the defining equation for A^{--} (3.11) and rewrite (4.18) as follows

$$\begin{aligned} &\omega \partial^{++} (\partial^+)^4 (\bar{\epsilon}\bar{\kappa})_{BC} \theta^{-B} \theta^{-C} A^{--} - \omega (\partial^+)^4 (\bar{\epsilon}\bar{\kappa})_{BC} \theta^{-B} \theta^{-C} \partial^{++} A^{--} \\ &= \nabla^{++} \Lambda_{\bar{Q}\bar{Q}} - \omega (\partial^+)^4 (\bar{\epsilon}\bar{\kappa})_{BC} \theta^{-B} \theta^{-C} \partial^{--} A^{++}. \end{aligned}$$

The second term in the last line vanishes due to the analyticity of A^{++} and the four derivatives ∂^+ while the first term is a compensating gauge transformation with parameter

$$\Lambda_{\bar{Q}\bar{Q}}(\theta^+) = \omega(\partial^+)^4 [(\bar{\epsilon}\bar{\kappa})_{BC} \theta^{-B} \theta^{-C} A^{--}]. \quad (4.19)$$

We emphasize that $\Lambda_{\bar{Q}\bar{Q}}$ is analytic, as it should be, due to projector $(\partial^+)^4$ in it. Therefore, the supersymmetry algebra closes on the connection A^{++}

$$[\delta(\kappa), \delta(\epsilon)] A^{++} = (\epsilon^{\beta B} \bar{\kappa}_B^{\dot{\beta}} - \kappa^{\beta B} \bar{\epsilon}_B^{\dot{\beta}}) \partial_{\beta \dot{\beta}} A^{++} + \nabla^{++} (\Lambda_{Q\bar{Q}} + \Lambda_{\bar{Q}Q}).$$

Next, let us show the closure of the algebra on the connection A_α^+ . Using (4.12) and (4.13) we find the commutator of the \bar{Q} and Q transformations

$$[\delta_{\bar{Q}}(\kappa), \delta_Q(\epsilon)] A_\alpha^+ = -(\epsilon^{+B} \bar{\kappa}_B^{\dot{\beta}}) \partial_{\dot{\beta}}^- A_\alpha^+ - \omega(\partial^+)^4 (\epsilon^{-B} \bar{\kappa}_{\dot{\alpha}B}) A^{--}. \quad (4.20)$$

Here A^{--} transforms by shifts of θ under Q supersymmetry, i.e. $\delta_Q(\epsilon) A^{--} = \epsilon^{\beta B} \partial_{\beta B} A^{--}$. This agrees with the definition of A^{--} in (3.11). In the first term on the right-hand side we complete the coordinate shift and then use the field equation (3.17) to get

$$\begin{aligned} -(\epsilon^{+B} \bar{\kappa}_B^{\dot{\beta}}) \partial_{\dot{\beta}}^- A_\alpha^+ &= (\epsilon^{\beta B} \bar{\kappa}_B^{\dot{\beta}}) \partial_{\beta \dot{\beta}} A_\alpha^+ - (\epsilon^{-B} \bar{\kappa}_B^{\dot{\beta}}) \partial_{\dot{\beta}}^+ A_\alpha^+ \\ &= (\epsilon^{\beta B} \bar{\kappa}_B^{\dot{\beta}}) \partial_{\beta \dot{\beta}} A_\alpha^+ - \nabla_\alpha^+ (\epsilon^{-B} \bar{\kappa}_B^{\dot{\beta}}) A_\beta^+ + \omega(\epsilon^{-B} \bar{\kappa}_{\dot{\alpha}B}) (\partial^+)^4 A^{--}. \end{aligned}$$

The first term in the second line is a translation, the second is a compensating gauge transformation with parameter (4.5) and the third cancels against a similar term in (4.20). We conclude that

$$[\delta_{\bar{Q}}(\kappa), \delta_Q(\epsilon)] A_\alpha^+ = \epsilon^{\beta B} \bar{\kappa}_B^{\dot{\beta}} \partial_{\beta \dot{\beta}} A_\alpha^+ + \nabla_\alpha^+ \Lambda_{Q\bar{Q}}. \quad (4.21)$$

Finally, using (4.13) and (4.16) we evaluate the commutator of two \bar{Q} transformations on the connection A_α^+ :

$$\begin{aligned} [\delta_{\bar{Q}}(\kappa), \delta_{\bar{Q}}(\epsilon)] A_\alpha^+ &= -\omega(\partial^+)^4 \left[(\bar{\epsilon}_{\dot{\alpha}C} \bar{\kappa}_B^{\dot{\beta}}) \theta^{+B} \theta^{-C} \partial_{\dot{\beta}}^- A^{--} + \bar{\kappa}_{\dot{\alpha}B} \theta^{-B} \delta_{\bar{Q}}(\epsilon) A^{--} \right] - (\epsilon \leftrightarrow \kappa) \\ &= \omega(\partial^+)^4 \left[(\bar{\epsilon} \bar{\kappa})_{BC} \theta^{-B} \theta^{-C} \nabla_\alpha^+ A^{--} - \partial^{--} (\bar{\epsilon} \bar{\kappa})_{BC} \theta^{-B} \theta^{-C} A_\beta^+ \right] \\ &= \nabla_\alpha^+ \Lambda_{\bar{Q}\bar{Q}}. \end{aligned} \quad (4.22)$$

This is again a compensating gauge transformation with the parameter (4.19). The second term in the second line of (4.22) drops out due to the projector $(\partial^+)^4$ and the analyticity of A_α^+ . So, the supersymmetry algebra closes on the connection A_α^+ as well,

$$[\delta(\kappa), \delta(\epsilon)] A_\alpha^+ = (\epsilon^{\beta B} \bar{\kappa}_B^{\dot{\beta}} - \kappa^{\beta B} \bar{\epsilon}_B^{\dot{\beta}}) \partial_{\beta \dot{\beta}} A_\alpha^+ + \nabla_\alpha^+ (\Lambda_{Q\bar{Q}} + \Lambda_{\bar{Q}Q}).$$

4.3.1 Closure of the \bar{Q} subalgebra off and on shell

We have checked that the full supersymmetry algebra is realized on the superconnections, modulo gauge transformations and the field equations (the kinematical constraint (3.5) and dynamical equation of motion (3.16)). The Q supersymmetry transformations (4.12) form an Abelian subalgebra off shell, i.e. the Q -half of the supersymmetry is explicit our

formalism. A natural question to ask is if the \bar{Q} -half also forms an Abelian subalgebra off shell. This will be important for us in [1] where we employ \bar{Q} transformations to construct the full stress-tensor supermultiplet starting from its chiral truncation. We use \bar{Q} in the form (4.13) and (4.14). In Sect. 4.3 we have already shown, without using the field equations, that two \bar{Q} anticommute on A^{++} up to a gauge transformation (4.19),

$$[\delta_{\bar{Q}}(\kappa), \delta_{\bar{Q}}(\epsilon)]A^{++} = \nabla^{++}\Lambda_{\bar{Q}\bar{Q}} \quad \text{off shell}.$$

However, on A_{α}^{+} the anticommutation (up to a gauge transformation) is valid only modulo the kinematical constraint (3.5). More precisely, we have

$$\begin{aligned} [\delta_{\bar{Q}}(\kappa), \delta_{\bar{Q}}(\epsilon)]A_{\alpha}^{+} &= \omega (\bar{\kappa}_A^{\dot{\beta}}\bar{\epsilon}_{\dot{\alpha}B} - \bar{\epsilon}_A^{\dot{\beta}}\bar{\kappa}_{\dot{\alpha}B})(\partial^{+})^4\theta^{-B} \int dv \left(\theta^{-A}W_{\dot{\beta}}^{+3}\frac{\delta}{\delta A^{++}} \right)(v) A^{-} \\ &\quad + \nabla_{\alpha}^{+}\Lambda_{\bar{Q}\bar{Q}} \quad \text{off shell}, \end{aligned}$$

where the curvature W_{α}^{+3} is defined in (3.4), the dependence on the dummy harmonic v is indicated explicitly. We plan to use this anticommutation relation repeatedly on the same superfields. This is possible because the field equations (3.5) and (3.16) transform into each other under \bar{Q} supersymmetry,

$$\begin{aligned} \bar{Q}_{\dot{\beta}}^B W_{\alpha}^{+3} &= \theta^{+B}\partial_{\dot{\beta}}^{-}W_{\alpha}^{+3} + \epsilon_{\alpha\dot{\beta}}\theta^{+B}(W^{++} - \omega \mathbb{W}^{++}) \\ \bar{Q}_{\dot{\beta}}^B(W^{++} - \omega \mathbb{W}^{++}) &= \theta^{+B}\partial_{\dot{\beta}}^{-}(W^{++} - \omega \mathbb{W}^{++}) - \omega(\partial^{+})^4 \int dv \left(\theta^{-B}W_{\dot{\beta}}^{+3}\frac{\delta}{\delta A^{++}} \right)(v) A^{-}. \end{aligned}$$

4.4 \bar{Q} supersymmetry and gauge transformations

The supersymmetry transformations ought to commute with the gauge transformations. This is evident for the Q -half (see (4.12)). The \bar{Q} -half is realized non-linearly and it mixes the two gauge connections, so the commutativity is not obvious. Here we prove that the \bar{Q} transformations (4.13), (4.14) commute with the gauge transformation (3.3),

$$[\delta_{\bar{Q}}, \delta_{\Lambda}]A^{++} = [\delta_{\bar{Q}}, \delta_{\Lambda}]A_{\alpha}^{+} = 0. \quad (4.23)$$

This result is crucial for us in the twin paper [1], where we consider chiral gauge-invariant operators in LHC superspace. There we use the \bar{Q} -variations to construct their non-chiral extensions. The commutativity (4.23) ensures that the variations are gauge invariant operators as well. We emphasize that the commutativity holds without the field equations.

We start with δA^{++} . Denoting $(\bar{\epsilon}^{\dot{\beta}} \cdot \theta^{+}) \equiv \bar{\epsilon}_B^{\dot{\beta}}\theta^{+B}$ and recalling (3.3) and (4.14), we find

$$\begin{aligned} \delta_{\Lambda}\delta_{\bar{Q}}A^{++} &= \delta_{\Lambda}((\bar{\epsilon}^{\dot{\beta}} \cdot \theta^{+})(\partial_{\dot{\beta}}^{-}A^{++} + A_{\dot{\beta}}^{+})) = (\bar{\epsilon}^{\dot{\beta}} \cdot \theta^{+})(\partial_{\dot{\beta}}^{-}\nabla^{++}\Lambda + \nabla_{\dot{\beta}}^{+}\Lambda) \\ &= (\bar{\epsilon}^{\dot{\beta}} \cdot \theta^{+})(\partial^{++}\partial_{\dot{\beta}}^{-}\Lambda + \partial_{\dot{\beta}}^{-}[A^{++}, \Lambda] + [A_{\dot{\beta}}^{+}, \Lambda]). \end{aligned}$$

The inverse sequence of variations leads to

$$\begin{aligned}
\delta_{\bar{Q}} \delta_{\Lambda} A^{++} &= \delta_{\bar{Q}} (\partial^{++} \Lambda + [A^{++}, \Lambda]) \\
&= \partial^{++} ((\bar{\epsilon}^{\dot{\beta}} \cdot \theta^+) \partial_{\dot{\beta}}^- \Lambda) + [(\bar{\epsilon}^{\dot{\beta}} \cdot \theta^+) (\partial_{\dot{\beta}}^- A^{++} + A_{\dot{\beta}}^+), \Lambda] + [A^{++}, (\bar{\epsilon}^{\dot{\beta}} \cdot \theta^+) \partial_{\dot{\beta}}^- \Lambda] \\
&= \delta_{\Lambda} \delta_{\bar{Q}} A^{++}.
\end{aligned}$$

Here we assume that the gauge parameter transforms under \bar{Q} supersymmetry as a chiral-analytic semi-superfield, $\delta_{\bar{Q}} \Lambda = (\bar{\epsilon}^{\dot{\beta}} \cdot \theta^+) \partial_{\dot{\beta}}^- \Lambda$ (cf. (4.1)). The first part of (4.23) is proven.

Now we turn to the variation of A_{α}^+ (recall (3.9)):

$$\begin{aligned}
\delta_{\Lambda} \delta_{\bar{Q}} A_{\alpha}^+ &= \delta_{\Lambda} \left((\bar{\epsilon}^{\dot{\beta}} \cdot \theta^+) \partial_{\dot{\beta}}^- A_{\alpha}^+ + \omega (\partial^+)^4 (\bar{\epsilon}_{\dot{\alpha}} \cdot \theta^-) A^{--} \right) \\
&= (\bar{\epsilon}^{\dot{\beta}} \cdot \theta^+) \partial_{\dot{\beta}}^- (\nabla_{\alpha}^+ \Lambda) + \omega (\partial^+)^4 (\bar{\epsilon}_{\dot{\alpha}} \cdot \theta^-) (\partial^{--} \Lambda + [A^{--}, \Lambda]) \\
&= (\bar{\epsilon}^{\dot{\beta}} \cdot \theta^+) \partial_{\dot{\beta}}^- (\partial_{\alpha}^+ \Lambda + [A_{\alpha}^+, \Lambda]) + \omega [(\partial^+)^4 ((\bar{\epsilon}_{\dot{\alpha}} \cdot \theta^-) A^{--}), \Lambda],
\end{aligned}$$

where we used the fact that $(\partial^+)^4 (\theta^{-B} \partial^{--} \Lambda) = 0$ due to the analyticity of the gauge parameter, $\partial^+ \Lambda = 0$. On the other hand,

$$\begin{aligned}
\delta_{\bar{Q}} \delta_{\Lambda} A_{\alpha}^+ &= \delta_{\bar{Q}} (\partial_{\alpha}^+ \Lambda + [A_{\alpha}^+, \Lambda]) \\
&= \partial_{\alpha}^+ (\bar{\epsilon}^{\dot{\beta}} \cdot \theta^+) \partial_{\dot{\beta}}^- \Lambda + [(\bar{\epsilon}^{\dot{\beta}} \cdot \theta^+) \partial_{\dot{\beta}}^- A_{\alpha}^+ + \omega (\partial^+)^4 (\bar{\epsilon}_{\dot{\alpha}} \cdot \theta^-) A^{--}, \Lambda] + [A_{\alpha}^+, (\bar{\epsilon}^{\dot{\beta}} \cdot \theta^+) \partial_{\dot{\beta}}^- \Lambda] \\
&= \delta_{\Lambda} \delta_{\bar{Q}} A_{\alpha}^+.
\end{aligned}$$

Thus the second equality in (4.23) is proven.

4.5 \bar{Q} supersymmetry in the WZ gauge

Here we examine the \bar{Q} supersymmetry transformations in the WZ gauge (3.36). We would like to show that the LHC formulae of the previous subsections reproduce the familiar transformation rules for the physical component fields. For simplicity, we restrict the discussion to the linearized (free) case. We demonstrate once more that the modification of the supersymmetry transformations (4.7) is necessary even in the free theory. In order to maintain the gauge we have to make a compensating gauge transformation (3.3) with parameter $\Lambda = -(\bar{\epsilon}_B^{\dot{\beta}} \theta^{+B}) \mathcal{A}_{\dot{\beta}}^-$ depending on the gauge field $\mathcal{A}_{\beta\dot{\beta}}$.

Let us start with the \bar{Q} transformations of A_{α}^+ in the self-dual theory (4.1). We use a shorthand notation for the composition of the two transformations $\delta = \delta_{\text{CS}+\Lambda}(\epsilon)$. We have

$$\delta A_{\alpha}^+ = \bar{\epsilon}_B^{\dot{\beta}} \theta^{+B} \partial_{\dot{\beta}}^- A_{\alpha}^+ + \partial_{\alpha}^+ \Lambda.$$

Inserting the WZ gauge (3.37), we deduce the transformation rules for the component fields \mathcal{A} and $\bar{\psi}$. It is clear that the gauge field \mathcal{A} does not transform. At level $(\theta^+)^1$ we get

$$\delta \bar{\psi}_{\dot{\alpha}B} = -\bar{\epsilon}_B^{\dot{\beta}} (\partial_{\dot{\beta}}^- \mathcal{A}_{\alpha}^+ - \partial_{\alpha}^+ \mathcal{A}_{\dot{\beta}}^-) = \bar{\epsilon}_{\dot{\alpha}B} F^{+-} + \bar{\epsilon}_B^{\dot{\beta}} \tilde{F}_{\dot{\alpha}\dot{\beta}}, \quad (4.24)$$

where $F^{\alpha\beta} = \partial^{\dot{\beta}(\alpha} \mathcal{A}_{\dot{\beta}}^{\beta)}$ and $\tilde{F}_{\dot{\alpha}\dot{\beta}} = \partial_{(\dot{\alpha}}^{\alpha} \mathcal{A}_{\dot{\beta})\alpha}$ are the two halves of the YM curvature. The \tilde{F} -term in the second relation is the expected transformation of the component $\bar{\psi}_{\dot{\alpha}B}$, while the F -term is unwanted. In the self-dual theory we have the field equation $F^{\alpha\beta} = 0$ (recall (3.18)) and the unwanted term vanishes. However, in the full theory the equation is modified, $F^{\alpha\beta} = \omega G^{\alpha\beta} \neq 0$ (recall (3.41)). We expect that this term is compensated by the modification (4.7). Indeed, in the linearized case and in the WZ gauge we have

$$\delta_Z A^{+\dot{\alpha}} = \omega(\partial^+)^4 \bar{\epsilon}_B^{\dot{\alpha}} \theta^{-B} A^{--} = -\frac{\omega}{2} \bar{\epsilon}_B^{\dot{\alpha}} \psi^{+B} - \omega \theta^{+B} \bar{\epsilon}_B^{\dot{\alpha}} G^{+-}.$$

The fermion term in this formula generates the expected transformation of the YM field,

$$\delta_Z \mathcal{A}^{\dot{\alpha}\alpha} = -\frac{\omega}{2} \bar{\epsilon}_A^{\dot{\alpha}} \psi^{\alpha A}, \quad (4.25)$$

while the G -term compensates the F -term in (4.24) due to the field equation and we find

$$\delta_{\text{CS}+\text{Z}+\Lambda} \bar{\psi}_{\dot{\alpha}B} = \bar{\epsilon}_B^{\dot{\beta}} \tilde{F}_{\dot{\alpha}\dot{\beta}}.$$

For the other prepotential A^{++} the \bar{Q} transformation rules are the same as in the self-dual theory, (4.2) and (4.14), so we have

$$\delta A^{++} = \bar{\epsilon}_B^{\dot{\beta}} \theta^{+B} [\partial_{\dot{\beta}}^- A^{++} + A_{\dot{\beta}}^+ - \mathcal{A}_{\dot{\beta}}^+].$$

Substituting the WZ gauges (3.36) and (3.37) in this formula, we find

$$\begin{aligned} \delta \phi_{AB} &= 2 \bar{\epsilon}_{[A}^{\dot{\beta}} \bar{\psi}_{B]\dot{\beta}} \\ \delta \psi_{\alpha}^A &= -2 \bar{\epsilon}_B^{\dot{\beta}} \partial_{\alpha\dot{\beta}} \phi^{AB} \\ \delta G^{\alpha\beta} &= \frac{1}{2} \bar{\epsilon}_B^{\dot{\beta}} \partial_{\dot{\beta}}^{(\alpha} \psi^{\beta)B}. \end{aligned}$$

Here we eliminated the auxiliary fields B^- and τ^{--} by means of the field equations (3.39).

The component field transformations found here are in agreement with those in [39]. In particular, the non-self-dual modification concerns only the gauge field (4.25).

5 Quantization

In this section we quantize the $\mathcal{N} = 4$ SYM theory with the classical action (3.25). In order to define propagators for the gauge connections we need to fix a gauge. In Sect. 5.1 we discuss the light-cone gauge. Then in Sect. 5.2 we derive the propagators for the dynamical superfields A^{++} and $A_{\dot{\alpha}}^+$ in this gauge, and after that we summarize the Feynman rules in Sect. 5.3. In Sect. 5.4 we give some simple examples of LHC supergraph calculations.

5.1 Light-cone gauge

We wish to implement the so-called “axial” or “CSW” gauge of [23], [27]. Translating to our notation, it is simply

$$\lambda^{\dot{\alpha}} A_{\dot{\alpha}}^+ = 0, \quad (5.1)$$

where $\lambda^{\dot{\alpha}}$ is an arbitrary but fixed commuting anti-chiral spinor.¹³ We prefer the name “light-cone” to the name “axial” used in [23] and other twistor papers. Indeed, remember that the ordinary Yang-Mills field $\mathcal{A}_{\alpha\dot{\alpha}}(x)$ lives in the connection $A_{\dot{\alpha}}^+$, see the first line in (3.39). Then the condition (5.1) restricted to this particular component becomes $\lambda_{\dot{\alpha}} u_{\alpha}^+ \mathcal{A}^{\dot{\alpha}\alpha}(x) = 0$. Here we see the light-like vector $n_{\alpha\dot{\alpha}} = \lambda_{\dot{\alpha}} u_{\alpha}^+$ typical for the light-cone gauge, rather than the non-null vector n^{μ} of the standard axial gauge [40].

An important advantage of the gauge (5.1) is that the cubic interaction term in the Chern-Simons Lagrangian (3.21) vanishes. Indeed, the general solution of (5.1) is $A_{\dot{\alpha}}^+ = \lambda_{\dot{\alpha}} A^+$, hence $A^{+\dot{\alpha}} A_{\dot{\alpha}}^+ = 0$. Thus, the connection $A_{\dot{\alpha}}^+$ becomes non-interacting but still propagating. Another advantage of this type of gauge is the decoupling of the ghosts [40].

One should make sure that the gauge (5.1) is possible. To answer this question, we make an Abelian gauge transformation and try to find a parameter Λ which can eliminate the projection in (5.1):

$$\lambda^{\dot{\alpha}} A_{\dot{\alpha}}^+ = \lambda_{\dot{\alpha}} \partial^{+\dot{\alpha}} \Lambda(x, \theta^+, u).$$

The operator $\lambda_{\dot{\alpha}} \partial^{+\dot{\alpha}}$ is invertible on the space of Λ , just like inverting a particular projection of the momentum. So the gauge (5.1) is possible under appropriate boundary conditions.

Let us now rewrite the gauge-fixing condition (5.1),

$$\lambda^{\dot{\alpha}} A_{\dot{\alpha}}^+ = 0 \quad (5.2)$$

and give it a new interpretation. We consider the parameter $\lambda^{\dot{\alpha}}$ as belonging to a new set of LH variables, this time for the second (anti-chiral) factor of the Lorentz group $SU(2)_L \times SU(2)_R$. These new harmonics satisfy the usual $SU(2)$ conditions (recall (2.5))

$$\lambda^{+\dot{\alpha}} \lambda_{\dot{\alpha}}^{\dot{-}} = 1, \quad (\lambda^{+\dot{\alpha}})^* = \lambda_{\dot{\alpha}}^{\dot{-}} \quad (5.3)$$

and are defined up to a $U(1)$ phase. To distinguish the harmonic charges of u_{α}^{\pm} and of $\lambda_{\dot{\alpha}}^{\pm}$ we put a dot above the latter (like the anti-chiral spinor indices $\dot{\alpha}$). The main difference between $\lambda_{\dot{\alpha}}^{\pm}$ and u_{α}^{\pm} is that the former are fixed parameters while the latter are coordinates. With the help of the two sets of harmonics we can project any Lorentz tensor onto its *light-cone components*. For example, the space-time coordinate $x^{\dot{\alpha}\alpha}$ can be decomposed into

$$x^{\dot{\alpha}\alpha} = -\lambda^{\dot{-}\dot{\alpha}} u^{+\alpha} x^{\dot{+}-} - \lambda^{+\dot{\alpha}} u^{-\alpha} x^{\dot{-}+} + \lambda^{\dot{-}\dot{\alpha}} u^{-\alpha} x^{\dot{+}+} + \lambda^{+\dot{\alpha}} u^{+\alpha} x^{\dot{-}-}, \quad (5.4)$$

where we see the *covariant light-cone projections*

$$x^{\pm\pm} = \lambda_{\dot{\alpha}}^{\pm} x^{\dot{\alpha}\alpha} u_{\alpha}^{\pm} \quad (5.5)$$

¹³In [26] the gauge is generalized by replacing $\lambda_{\dot{\alpha}}$ by a full ‘reference super-twistor’ $Z_* = (\mu_{\alpha}, \lambda_{\dot{\alpha}}, \xi^A)$. For our purposes the simpler form (5.1) suffices. Moreover, it seems unnatural to use the fermion ξ in Z_* to fix a gauge for the *bosonic* semi-superfields $A^{++}, A_{\dot{\alpha}}^+$.

obtained by using the four light-like vectors $n_\mu^{\pm\pm} = \lambda_\alpha^\pm (\sigma_\mu)^{\dot{\alpha}\alpha} u_\alpha^\pm$. If we make some standard choice of the harmonics, e.g., $\lambda_\alpha^+ = (1, 0)$, $\lambda_\alpha^- = (0, 1)$, etc., the projections (5.5) become the usual light-cone projections.

5.2 Green's functions

Now we proceed to the derivation of the propagators for the dynamical superfields A^{++} and A_α^+ . To simplify the derivation, we shall adopt the approach of [23], which is to treat the bilinear term in the second line of (3.26), coming from the Zupnik Lagrangian (3.23), as a ‘bivalent vertex’ (see also the comments in Sect. 5.4). Then the relevant kinetic terms are contained in the first line of (3.26), coming from the Chern-Simons Lagrangian (3.21). Adding sources for the two dynamical superfields, we obtain

$$L_{\text{kin}} = A^{++} \partial^{+\dot{\alpha}} A_\alpha^+ - \frac{1}{2} A^{+\dot{\alpha}} \partial^{++} A_\alpha^+ - A^{++} J^{++} - A^{+\dot{\alpha}} J_\alpha^{+3}, \quad (5.6)$$

where the factor of π has been introduced for convenience. Like the super-connections, the sources are chiral-analytic semi-superfields, $J(x, \theta^+, u)$. Due to the gauge invariance (3.3), the sources satisfy the conservation condition

$$\partial^{++} J^{++} + \partial^{+\dot{\alpha}} J_\alpha^{+3} = 0. \quad (5.7)$$

The linearized field equations are

$$\delta/\delta A^{++} : \quad \partial^{+\dot{\alpha}} A_\alpha^+ = J^{++}, \quad \delta/\delta A^{+\dot{\alpha}} : \quad \partial_\alpha^+ A^{++} - \partial^{++} A_\alpha^+ = J_\alpha^{+3}. \quad (5.8)$$

They are solved in terms of Green's functions:

$$A^{++}(1) = \int_2 \langle A^{++}(1) A^{++}(2) \rangle J^{++}(2) + \int_2 \langle A^{++}(1) A^{+\dot{\alpha}}(2) \rangle J_\alpha^{+3}(2) \quad (5.9)$$

$$A_\alpha^+(1) = \int_2 \langle A_\alpha^+(1) A^{++}(2) \rangle J^{++}(2) + \int_2 \langle A_\alpha^+(1) A^{+\dot{\beta}}(2) \rangle J_{\dot{\beta}}^{+3}(2). \quad (5.10)$$

Here we adopt the shorthand notations $A^{++}(k) = A^{++}(x_k, \theta_k^+, u_k)$ etc., and we omit the integration measure $d^4 x_2 du_2 d^4 \theta_2^+$. The corresponding Green's function equations are obtained by substituting (5.9), (5.10) in (5.8), shifting the derivatives from point 1 to 2, integrating by parts and using (5.7).

Below we show that a solution of the Green's function equations and of the gauge fixing condition (5.2) is given by the set of propagators¹⁴

$$\langle A_\alpha^+(1) A^{+\dot{\beta}}(2) \rangle = 0; \quad (5.11)$$

$$\langle A^{++}(x, \theta^+, u_1) A^{++}(0, 0, u_2) \rangle = \frac{1}{\pi} \delta^2(x^{++}) \delta(u_1, u_2) \delta^{(4)}(\theta^+); \quad (5.12)$$

$$\langle A_\alpha^+(x, \theta^+, u_1) A^{++}(0, 0, u_2) \rangle = -\langle A^{++}(1) A_\alpha^+(2) \rangle = \frac{1}{\pi} \frac{\lambda_{\dot{\alpha}}^-}{x^{++}} \delta^2(x^{++}) \delta(u_1, u_2) \delta^{(4)}(\theta^+), \quad (5.13)$$

¹⁴Similar propagators, in momentum space and for the bosonic YM theory, appear in [41].

where we have used translation invariance to set $x_2 = \theta_2 = 0$. We omit for simplicity the color indices δ_{ab} of the propagators. The bosonic delta functions are defined in App. A.2, A.3. Notice that we do not specify if the projections (5.5) are made with the harmonic u_1 or u_2 , in view of the harmonic delta functions above.

Let us check that this is a solution of (5.8). The gauge fixing condition (5.2) is obviously satisfied by (5.13). From (A.20), (A.21), (A.22) and (A.23) we get

$$\begin{aligned}\partial^{+\dot{\alpha}} \langle A_{\dot{\alpha}}^+(1) A^{++}(2) \rangle &= \delta^2(x^{\dot{-}+}) \delta^2(x^{\dot{+}+}) \delta(u_1, u_2) \delta^{(4)}(\theta^+) \\ &= \delta^4(x) \delta(u_1, u_2) \delta^{(4)}(\theta^+); \end{aligned} \quad (5.14)$$

$$\begin{aligned}\partial^{++} \langle A_{\dot{\alpha}}^+(1) A^{++}(2) \rangle &= \frac{1}{\pi} \lambda_{\dot{\alpha}}^{\dot{-}} \partial_{x^{\dot{-}-}} \delta^2(x^{\dot{+}+}) \delta(u_1, u_2) \delta^{(4)}(\theta^+) + \\ &+ \frac{1}{\pi} \frac{\lambda_{\dot{\alpha}}^{\dot{-}}}{x^{\dot{-}+}} \delta^2(x^{\dot{+}+}) \partial^{++} \delta(u_1, u_2) \delta^{(4)}(\theta^+); \end{aligned} \quad (5.15)$$

$$\partial_{\dot{\alpha}}^+ \langle A^{++}(1) A^{++}(2) \rangle = \frac{1}{\pi} \lambda_{\dot{\alpha}}^{\dot{-}} \partial_{x^{\dot{-}-}} \delta^2(x^{\dot{+}+}) \delta(u_1, u_2) \delta^{(4)}(\theta^+). \quad (5.16)$$

We start with the first equation in (5.8) and substitute (5.10), (5.13) and (5.11) into it. Taking into account (5.14) we get

$$\partial^{+\dot{\alpha}} A_{\dot{\alpha}}^+(1) = \int_2 \partial^{+\dot{\alpha}} \langle A_{\dot{\alpha}}^+(1) A^{++}(2) \rangle J^{++}(2) = J^{++}(1).$$

The second equation in (5.8) is more subtle. We substitute (5.9), (5.10) and (5.11) in it:

$$\int_2 \left[\partial_{\dot{\alpha}}^+ \langle A^{++}(1) A^{++}(2) \rangle J^{++}(2) + \partial_{\dot{\alpha}}^+ \langle A^{++}(1) A^{+\dot{\beta}}(2) \rangle J_{\dot{\beta}}^{+3}(2) - \partial^{++} \langle A_{\dot{\alpha}}^+(1) A^{++}(2) \rangle J^{++}(2) \right]. \quad (5.17)$$

We need to show that this expression equals $J_{\dot{\alpha}}^{+3}$. In the second term we use translation invariance to bring $\partial_{\dot{\alpha}}^+$ from the first point of the propagator $\langle A^{++}(1) A^{+\dot{\beta}}(2) \rangle$ to the second point. Then we integrate $\partial_{\dot{\alpha}}^+$ by parts onto the source and use its conservation (5.7),

$$\partial^{+\dot{\alpha}} J_{\dot{\beta}}^{+3} = \partial_{\dot{\beta}}^+ J^{+3\dot{\alpha}} + \delta_{\dot{\beta}}^{\dot{\alpha}} (\partial^{+\dot{\gamma}} J_{\dot{\gamma}}^{+3}) = \partial_{\dot{\beta}}^+ J^{+3\dot{\alpha}} - \delta_{\dot{\beta}}^{\dot{\alpha}} \partial^{++} J^{++}.$$

Thus in view of (5.14) the second term in (5.17) equals

$$\begin{aligned} \int_2 \partial_{\dot{\beta}}^+ \langle A^{++}(1) A^{+\dot{\beta}}(2) \rangle J^{+3\dot{\alpha}}(2) - \int_2 \langle A^{++}(1) A^{+\dot{\alpha}}(2) \rangle \partial^{++} J^{++}(2) &= \\ &= J^{+3\dot{\alpha}}(1) - \int_2 \langle A^{++}(1) A^{+\dot{\alpha}}(2) \rangle \partial^{++} J^{++}(2). \end{aligned} \quad (5.18)$$

We substitute (5.15) in the third term in (5.17), identify the first term of (5.15) with (5.16), and use (A.12) to integrate ∂^{++} in the second term of (5.15) by parts onto the source:

$$- \int_2 \left[\partial_{\dot{\alpha}}^+ \langle A^{++}(1) A^{++}(2) \rangle J^{++}(2) + \langle A_{\dot{\alpha}}^+(1) A^{++}(2) \rangle \partial^{++} J^{++}(2) \right]. \quad (5.19)$$

The first term in (5.19) cancels out the first term in (5.17). Since $\langle A_{\dot{\alpha}}^+(1) A^{++}(2) \rangle = -\langle A^{++}(1) A_{\dot{\alpha}}^+(2) \rangle$, the second term in (5.19) cancels out the second term in (5.18).

This concludes the verification of the propagators (5.11)–(5.13) in the gauge (5.2).

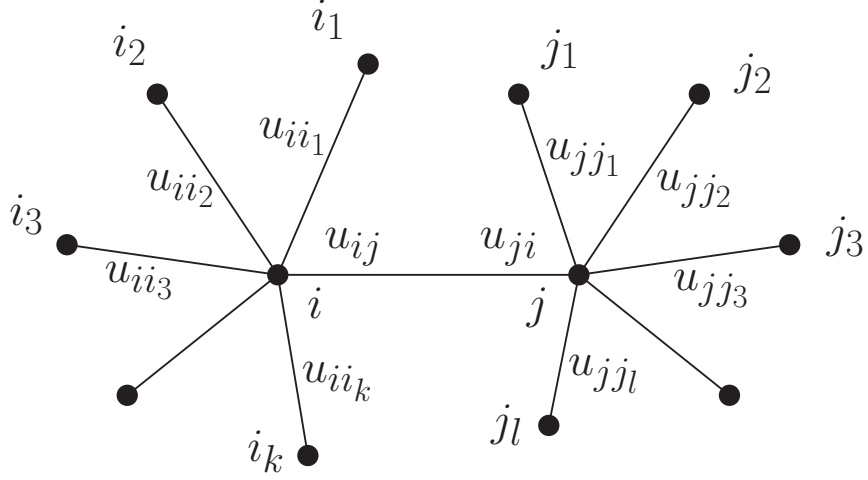


Figure 1: Part of a supergraph. The two interaction vertices i and j of valences k and l are connected by a propagator ij . Two sets of LHs $u_{ij}, u_{ii_1}, \dots, u_{ii_k}$ and $u_{ji}, u_{jj_1}, \dots, u_{jj_l}$ are associated with the vertices. The harmonic delta function in the propagator identifies $u_{ij} = u_{ji}$. The vertices i_1, i_2, \dots, i_k and j_1, j_2, \dots, j_l can be either external or interaction.

5.3 Feynman rules

The action (3.25), the light-cone gauge condition (5.2) and the propagators (5.12)–(5.11) lead to simple Feynman rules:

- To each internal (interaction) vertex i connected with the vertices j_1, j_2, \dots, j_k we assign a set of LHs $u_{ij_1}, u_{ij_2}, \dots, u_{ij_k}$.
- A LH integral and factor are attached to the vertex, namely

$$\omega (-1)^k \int \frac{\prod_{l=1}^k du_{ij_l}}{(u_{ij_1}^+ u_{ij_2}^+)(u_{ij_2}^+ u_{ij_3}^+) \dots (u_{ij_k}^+ u_{ij_1}^+)}.$$

- An integration $\int d^4x_i d^8\theta_i$ is associated with each internal vertex i .
- The color structure at the vertex is $\text{tr}(T^{a_{j_1}} \dots T^{a_{j_k}})$. The propagators are diagonal in the color space.
- The superfield A_{α}^+ can appear only at external vertices, but not at interaction vertices.
- We assign $\frac{1}{\pi} \delta^2(x_{ij}^{++}) \delta^4(\theta_{ij}^+) \delta(u_{ij}, u_{ji})$ to each propagator $\langle A^{++}(i) A^{++}(j) \rangle$ connecting vertices i and j . If vertex i is external and $A_{\alpha}^+(i)$ is present there, an additional factor $\lambda_{\alpha}^- / x_{ij}^{-+}$ appears.
- All the Grassmann integrals are done with the help of $\delta^4(\theta_{ij}^+)$ from the propagators.

- Using the harmonic delta functions $\delta(u_{ij}, u_{ji})$ from the propagators, half of the harmonic integrations can be done immediately leading to the identification $u_{ij} = u_{ji}$. The other half of the harmonic integrals are lifted with the help of the bosonic $\delta^2(x_{ij}^{++})$ from the propagators.

These Feynman rules prove to be extremely useful in the calculation of correlation functions of gauge invariant operators. The main reason is that the Born-level graphs do not contain interaction vertices, and so each relevant graph is a rational function. In [11] we used the twistor version of the Feynman rules to calculate the multipoint correlators of the chiral truncation of the stress-tensor supermultiplet. In [1] we use the LHC Feynman rules for the study of the multipoint correlation functions of the full (non-chiral) stress-tensor supermultiplet. We illustrate the Feynman rules in the next Section on the simple example of the classical and quantum corrections to the propagators.

5.4 Full propagators

Above we found the set of propagators using the quadratic Lagrangian (5.6). We recall that there is one more quadratic term in the full Lagrangian coming from L_Z (see (3.26)). Following [23], we chose to treat it as an interaction vertex. This is useful because the quadratic part of L_{CS} supplies rather simple propagators which are very convenient in Feynman diagram calculations. The quadratic term in L_Z is rather different from the quadratic part of L_{CS} since it is non-local in the harmonics and it contains four extra Grassmann integrations.

Still, we can ask the question how the extended quadratic Lagrangian modifies the form of the propagators. Instead of repeating the entire derivation of the previous subsection, it is easier to calculate the corrections to the ‘bare’ propagators (5.12)–(5.11) inserting into them the bivalent vertices from L_Z . Counting the Grassmann degrees of the relevant Feynman graphs we conclude that each insertion of the quadratic vertex from L_Z lowers the Grassmann degree by 4 units. Since the ‘bare’ propagators have Grassmann degree 4, then only a single insertion can give rise to a non-zero correction and the result does not depend on θ . A straightforward calculation yields the corrections

$$\begin{aligned}
\langle A^{++}(x, \theta^+, u_1) A^{++}(0, 0, u_2) \rangle_{\text{corr}} &= -\frac{\omega}{4\pi^2} \frac{(u_1^+ u_2^+)}{(u_1^- u_2^-)} \\
\langle A_\alpha^+(x, \theta^+, u_1) A^{++}(0, 0, u_2) \rangle_{\text{corr}} &= \frac{\omega}{4\pi^2} \lambda_{\dot{\alpha}} \frac{(u_1^+ u_2^+)}{(x^- u_2^-)} \\
\langle A_\alpha^+(x, \theta^+, u_1) A_\beta^+(0, 0, u_2) \rangle_{\text{corr}} &= \frac{\omega}{4\pi^2} \lambda_{\dot{\alpha}} \lambda_{\dot{\beta}} \frac{(u_1^+ u_2^+)(u_1^- u_2^-)}{(x^- u_1^-)(x^- u_2^-)}, \tag{5.20}
\end{aligned}$$

which are to be added to the bare propagators (5.12)–(5.11). These full propagators take into account the quadratic vertex from L_Z . If we use the full propagators instead of the bare ones, then all interaction vertices will be at least cubic in the fields. Following the

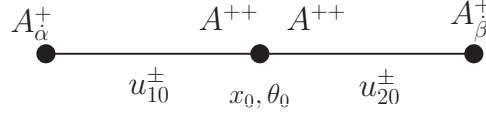


Figure 2: Bivalent vertex correction to the bare propagator $\langle A_\alpha^+(1)A_\beta^+(2) \rangle$.

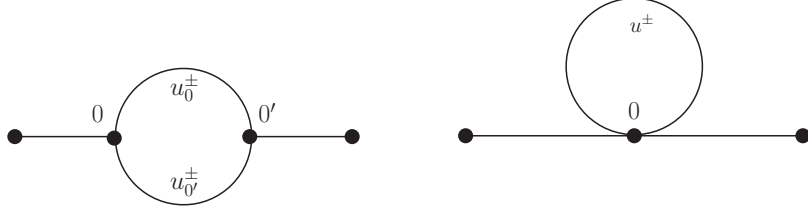


Figure 3: Graphs contributing to the two-point function $\langle A^{++}(1)A^{++}(2) \rangle$ at order $O(g^2)$.

line of Sect. 5.2, we have checked that the corrected propagators solve the analogs of the field equations (5.8) corresponding to the extended quadratic Lagrangian.

As an illustration, we calculate (5.20) explicitly. The insertion of a bivalent vertex at point 0 is depicted in Figure 2. It contributes

$$\begin{aligned} \langle A_\alpha^+(1)A_\beta^+(2) \rangle_{\text{corr}} &= -\omega \int d^8\theta_0 d^4x_0 \int \frac{du_{10}du_{20}}{(u_{10}^+u_{20}^+)^2} \langle A_\alpha^+(x_1, \theta_1^+, u_1) A^{++}(x_0, \theta_0 u_{10}^+, u_{10}) \rangle \\ &\quad \times \langle A^{++}(x_0, \theta_0 u_{20}^+, u_{20}) A_\beta^+(x_2, \theta_2^+, u_2) \rangle. \end{aligned}$$

Substituting the explicit expressions for the propagators (5.13) we obtain

$$\begin{aligned} \langle A_\alpha^+(1)A_\beta^+(2) \rangle_{\text{corr}} &= \frac{\omega}{\pi^2} \int d^8\theta_0 d^4x_0 \int \frac{du_{10}du_{20}}{(u_{10}^+u_{20}^+)^2} \frac{\lambda_\alpha^\dot{-}}{(x_{10}^+u_1^+)} \delta(u_1, u_{10}) \delta^4(\theta_{10}u_1^+) \delta^2(x_{10}^+u_1^+) \times \\ &\quad \frac{\lambda_\beta^\dot{-}}{(x_{20}^+u_2^+)} \delta(u_2, u_{20}) \delta^4(\theta_{20}u_2^+) \delta^2(x_{20}^+u_2^+) = \frac{\omega}{\pi^2} (u_1^+u_2^+)^2 \lambda_\alpha^\dot{-} \lambda_\beta^\dot{-} \int d^4x_0 \frac{\delta^2(x_{10}^+u_1^+) \delta^2(x_{20}^+u_2^+)}{(x_{10}^+u_1^+)(x_{20}^+u_2^+)}. \end{aligned}$$

On the support of the delta functions $(x_{10}^+u_1^+) = -\frac{(x_{12}^+u_2^-)}{(u_1^-u_2^-)}$ and $(x_{20}^+u_2^+) = -\frac{(x_{12}^+u_1^-)}{(u_1^-u_2^-)}$. The four delta functions make the integration over x_0 trivial and we obtain (5.20).

In what follows we prefer to deal with the bare propagators (5.12)–(5.11) and to consider the quadratic vertex L_Z as an interaction.

As another simple illustration of the Feynman rules from Sect. 5.3 let us show that the first and the second quantum corrections (orders $O(g^2)$ and $O(g^4)$) to the two-point function $\langle A^{++}(1)A^{++}(2) \rangle$ vanish. There are two potentially contributing Feynman graphs depicted in Figure 3. The blobs denote the integration $\int d^4x d^8\theta$. Both graphs vanish,

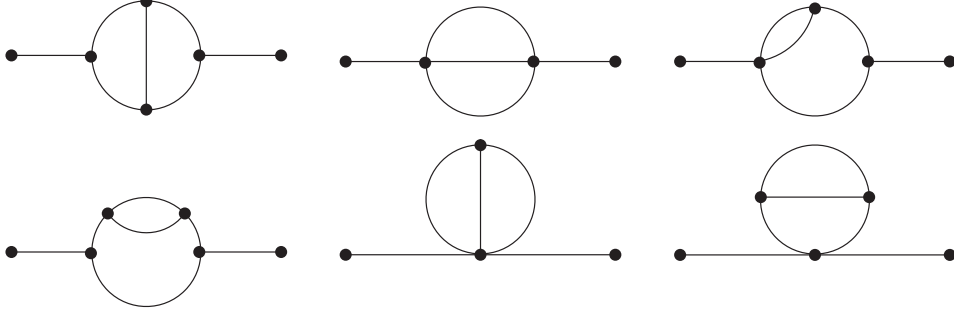


Figure 4: Graphs contributing to the two-point function $\langle A^{++}(1)A^{++}(2) \rangle$ at order $O(g^4)$.

indeed. The loop in the first graph contains

$$\frac{1}{(u_0^+ u_{0'}^+)^2} \delta^4(\theta_{00'} u_0^+) \delta^4(\theta_{00'} u_{0'}^+) \delta^2(x_{00'}^+ u_0^+) \delta^2(x_{00'}^+ u_{0'}^+) = (\theta_{00'})^8 (u_0^+ u_{0'}^+)^2 \delta^2(x_{00'}^+ u_0^+) \delta^2(x_{00'}^+ u_{0'}^+) = 0.$$

The second graph contains a self-closed line which is proportional to $\delta^4((\theta_0 - \theta_0)u^+) = 0$.

The LHC supergraphs of order $O(g^4)$ contributing to the two-point function are depicted in Figure 4. All of them vanish. Indeed, the formal counting of the Grassmann degree of the first graph gives -4 , i.e. it has to be zero. The remaining graphs contain the one-loop subgraph from Figure 3. It vanishes as we have already explained above, so the whole graph containing it vanishes as well.

At higher loop orders there are graphs that do not vanish, at least not for obvious reasons. The above calculation is presented just for illustration purposes. Much more useful applications of the LHC supergraphs are worked out in [1] where we deal with correlation functions of composite operators.

6 Analogy with $\mathcal{N} = 2$ harmonic superspace

$\mathcal{N} = 4$ SYM can be formulated in terms of $\mathcal{N} = 2$ superfields in harmonic superspace [4, 5]. The main ingredient there are the harmonic variables on $SU(2)$, but for the R-symmetry group rather than half of the Lorentz group as in this paper. Like the formulation in LHC superspace, the $\mathcal{N} = 2$ formulation has only half of the $\mathcal{N} = 4$ supersymmetry manifest, but it is a different half: there one has 2 Q and 2 \bar{Q} manifest supersymmetries, here we have all 4 Q 's but no \bar{Q} 's. Still, both the formulation and the way the non-manifest half of $\mathcal{N} = 4$ supersymmetry is realized are strikingly similar in the two harmonic superspaces.

Here we give a very brief summary of the $\mathcal{N} = 2$ formulation, referring the reader to [5] for more detail. The notations are modified in order to make the similarity with the LHC formulation more transparent. We make use of Grassmann (or G-)analytic superfields

$$\Phi(z_{\text{an}}^{a'a}, \theta^{+a}, \bar{\theta}^{+a'}, u), \quad \text{where } \theta^{+a} = u_{\alpha}^{+} \theta^{\alpha a}, \quad \bar{\theta}^{+a'} = u_{\alpha}^{+} \bar{\theta}^{\alpha a'}. \quad (6.1)$$

Here $\alpha = 1, 2$ is an index of the R-symmetry group $SU(2)$ and $a, a' = 1, 2$ are indices of the Euclidean Lorentz group $SO(4) \sim SU(2)_L \times SU(2)_R$. The space-time variable $z^{a'a}$ (to be distinguished from $x^{\dot{\alpha}\alpha}$ in (2.15)) is defined in the analytic basis

$$z_{\text{an}}^{a'a} = z^{a'a} + \bar{\theta}^{+a'} \theta^{-a} + \bar{\theta}^{-a'} \theta^{+a}. \quad (6.2)$$

From the point of view of $\mathcal{N} = 4$ supersymmetry these are ‘semi-superfields’ because they do not depend on the other half of the Grassmann variables, $\theta^{\dot{\alpha}a}, \bar{\theta}^{\dot{\alpha}a'}$ where $\dot{\alpha} = 1, 2$ completes the $SU(4)$ index $A = (\alpha, \dot{\alpha})$. The G-analytic superfields (6.1) depend only on a quarter of the $\mathcal{N} = 4$ odd variables, hence the $\mathcal{N} > 2$ half of the supersymmetry will be realized non-linearly and on shell.

We make use of two basic superfields

$$A_{\dot{\alpha}}^+ = A_{\dot{\alpha}}^+(z_{\text{an}}, \theta^+, \bar{\theta}^+, u), \quad A^{++} = A^{++}(z_{\text{an}}, \theta^+, \bar{\theta}^+, u). \quad (6.3)$$

The first of them describes the $\mathcal{N} = 2$ hypermultiplet (matter), with an additional index $\dot{\alpha} = 1, 2$ of the automorphism Pauli-Gursey group $SU(2)$. The second is the gauge connection (prepotential) for the supersymmetrized harmonic derivative

$$D^{++} = \partial^{++} - 2\theta^{+a}(\partial_{z_{\text{an}}})_{aa'}\bar{\theta}^{+a'} \quad \Rightarrow \quad \nabla^{++} = D^{++} + A^{++}, \quad (6.4)$$

with respect to a gauge group with an analytic parameter

$$\delta A_{\dot{\alpha}}^+ = [A_{\dot{\alpha}}^+, \Lambda], \quad \delta A^{++} = D^{++}\Lambda + [A^{++}, \Lambda], \quad \Lambda = \Lambda(z_{\text{an}}, \theta^+, \bar{\theta}^+, u). \quad (6.5)$$

Note that the HM $A_{\dot{\alpha}}^+$ now transforms as a matter field, not as a gauge field as in (3.3). The reason is the absence of a coordinate like $x_{\dot{\alpha}}^-$ and the corresponding derivative $\partial_{\dot{\alpha}}^+$, whose covariantization requires the gauge field in (3.2). Another difference is the space-time derivative term in the harmonic derivative D^{++} due to the change of basis (6.2).

6.1 Action

The $\mathcal{N} = 4$ SYM action for the superfields (6.3) consists of two terms,

$$S_{\mathcal{N}=4} = S_{\text{HM}} + S_Z. \quad (6.6)$$

The first involves only the HM $A_{\dot{\alpha}}^+$, the second depends only on A^{++} and is the original Zupnik’s action [12]. In the linearized (Abelian) theory the action (6.6) has the form

$$S_{\mathcal{N}=4} = -\frac{1}{2} \int d^4 z_{\text{an}} du d^2 \theta^+ d^2 \bar{\theta}^+ A^{+\dot{\alpha}} D^{++} A_{\dot{\alpha}}^+ - \frac{\omega}{2} \int d^4 z d^4 \theta d^4 \bar{\theta} du_1 du_2 \frac{A^{++}(z_{1\text{an}}, \theta_1^+, \bar{\theta}_1^+, u_1) A^{++}(z_{2\text{an}}, \theta_2^+, \bar{\theta}_2^+, u_2)}{(u_1^+ u_2^+)^2} + O(g), \quad (6.7)$$

where $\theta_i^+ = \theta^\alpha(u_i)_\alpha^+$, $\bar{\theta}_i^+ = \bar{\theta}^\alpha(u_i)_\alpha^+$ and $z_{i\text{an}} = z + \bar{\theta}_i^+ \theta_i^- + \bar{\theta}_i^- \theta_i^+$, with $i = 1, 2$. Note that the first term involves a G-analytic superspace integral, while the second has a full $\mathcal{N} = 2$ superspace integral.

In the interacting (non-Abelian) theory the action (6.7) becomes

$$S_{\mathcal{N}=4} = \int d^4 z_{\text{an}} du d^2 \theta^+ d^2 \bar{\theta}^+ \text{tr} \left(-\frac{1}{2} A^{+\dot{\alpha}} D^{++} A_{\dot{\alpha}}^+ + A^{++} A^{+\dot{\alpha}} A_{\dot{\alpha}}^+ \right) + \omega \int d^4 z d^4 \theta d^4 \bar{\theta} \log \det \left(\frac{1}{D^{++}} \nabla^{++} \right). \quad (6.8)$$

Here the HM action has an additional cubic term, which is the standard gauge-matter coupling. Compared to the Chern-Simons action (3.21), the only difference is the absence of the mixed kinetic term $A^{++} \partial^{+\dot{\alpha}} A_{\dot{\alpha}}^+$. This is again due to the absence of a coordinate like $x_{\dot{\alpha}}^-$ and the corresponding derivative $\partial_{\dot{\alpha}}^+$ in the $\mathcal{N} = 2$ formalism. The non-polynomial original Zupnik's term is almost identical with the LHC action (3.22), (3.23). The only difference is the Grassmann shift of the coordinates $z_{i\text{an}}$ in (6.7).

6.2 Equations of motion and component field content

The linearized field equations following from (6.7) are

$$\frac{\delta S_{\mathcal{N}=4}}{\delta A^{+\dot{\alpha}}} : \quad D^{++} A_{\dot{\alpha}}^+ = 0 \quad (6.9)$$

$$\frac{\delta S_{\mathcal{N}=4}}{\delta A^{++}} : \quad (D^+)^4 A^{--} = 0 \quad (6.10)$$

where A^{--} is the non-analytic connection for the harmonic derivative D^{--} . It is given by the non-polynomial and non-local (in the harmonic space) expression

$$A^{--}(z, \theta, \bar{\theta}, u) = \int du_2 \frac{A^{++}(z_{2\text{an}}, \theta_2^+, \bar{\theta}_2^+, u_2)}{(u^+ u_2^+)^2} + O(g). \quad (6.11)$$

The role of the supersymmetric spinor derivatives $(D^+)^4 = (D_a^+)^2 (\bar{D}_{a'}^+)^2$ in (6.10) is to project A^{--} onto the G-analytic subspace. These equations of motion are to be compared with the (linearized) LHC ones, (3.5) and (3.17), respectively. The main difference is that the field equations (6.9), (6.10) are decoupled, once again due to the absence of $\partial_{\dot{\alpha}}^+$.

The component field content of the gauge prepotential is revealed in the WZ gauge

$$A^{++} = (\theta^+)^2 \phi(z) + (\bar{\theta}^+)^2 \bar{\phi}(z) + \bar{\theta}^{+a} \theta^{+a'} \mathcal{A}_{aa'}(z) + (\bar{\theta}^+)^2 \theta^{+a} \psi_a^-(z) + (\theta^+)^2 \bar{\theta}^{+a'} \bar{\psi}_{a'}^-(z) + 3(\theta^+)^2 (\bar{\theta}^+)^2 u^{-a} u^{-\beta} G_{\alpha\beta}(z). \quad (6.12)$$

Here we see the full content of the $\mathcal{N} = 2$ *off-shell* vector multiplet, including the triplet of *auxiliary fields* $G_{\alpha\beta}$. Inserting (6.12) in the second field equation (6.10), we obtain the free equations of motion for the component fields. In particular, the auxiliary field satisfies the homogeneous equation

$$G_{\alpha\beta} = 0,$$

in contrast with the inhomogeneous equation $\omega G = F$ following from the action (3.41). This time the YM equation is obtained from (6.10) directly in the second-order form

$$\partial_z^\mu F_{\mu\nu}(z) = 0.$$

The first field equation (6.9) determines the content of the *on-shell* $\mathcal{N} = 2$ HM:

$$A_\alpha^+ = f_\alpha^\alpha(z)u_\alpha^+ + \theta^{+a}\chi_{a\dot{\alpha}}(z) + \bar{\theta}^{+a'}\bar{\chi}_{a'\dot{\alpha}}(z) + 2\theta^{+a}\bar{\theta}^{+a'}\partial_{aa'}f_\alpha^\alpha(z)u_\alpha^-, \quad (6.13)$$

where the physical fields (the four scalars f_α^α and the two Dirac fermions $\chi_{\dot{\alpha}}, \bar{\chi}_{\dot{\alpha}}$) satisfy their free equations of motion

$$\square_z f_\alpha^\alpha = \partial_z^{a'a}\chi_{a\dot{\alpha}} = 0.$$

We observe that the $\mathcal{N} = 4$ vector multiplet is split in two halves. The prepotential A^{++} contains 2 of the 6 physical scalars, the gauge field and half of the gluinos (the triplet of auxiliary fields G vanishes on shell). The HM A_α^+ contains the remaining 4 scalars and the other half of the gluinos.

6.3 Realization of the hidden $\mathcal{N} > 2$ supersymmetry

The two extra supersymmetries are realized in terms of the gauge and matter superfields as follows (see [5] and also [42] for a $d = 6$ harmonic superspace version):

$$\delta A^{++} = (\epsilon^{\dot{\alpha}a}\theta_a^+ + \bar{\epsilon}_{a'}^{\dot{\alpha}}\bar{\theta}^{+a'})A_\alpha^+, \quad \delta A_\alpha^+ = \omega(D^+)^4[(\epsilon_\alpha^a\theta_a^- + \bar{\epsilon}_{a'}^{\dot{\alpha}}\bar{\theta}^{-a'})A^{--}], \quad (6.14)$$

where ϵ_α^a and $\bar{\epsilon}_{a'}^{\dot{\alpha}} \equiv \overline{(\epsilon_\alpha^a)}$ are the relevant Grassmann parameters. The variation of A_α^+ is non-linear and non-local in the harmonic space because of the presence of A^{--} in it (see (3.28)). The supersymmetry algebra closes modulo field equations and compensating gauge transformations, as usual for on-shell supersymmetry.

Note the similarity with the realization of the hidden \bar{Q} supersymmetry in (4.13), (4.14). The absence of space-time derivative terms in (6.14) is due to the fact that now $\partial_x A = 0$.

6.4 Interpretation as eight-dimensional SYM

We have shown the remarkable similarity of the two formulations, up to a few details. A possible interpretation¹⁵ is that both are obtained from a master SYM theory in eight-dimensional Euclidean superspace enhanced with a harmonic two-sphere,

$$R_{x,z}^8 \times S_u^2,$$

having 8 odd coordinates $(\theta^{\alpha a}, \bar{\theta}^{\alpha a'})$. The full $d = 8$ superspace has twice as many odd coordinates, the missing ones being $(\tilde{\theta}^{\dot{\alpha} a}, \tilde{\bar{\theta}}^{\dot{\alpha} a'})$. Here the $d = 8$ Lorentz group $SO(8)$ is broken

¹⁵We are grateful to Evgeny Ivanov for a discussion on this point.

down to $SO(4) \times SO(4) \sim [SU(2)]^4$, and one of the $SU(2)$ factors is ‘harmonized’. On this superspace we consider the analytic superfields $A^{++}(x, z, u; \theta^+, \bar{\theta}^+)$ and $A_{\dot{\alpha}}^+(x, z, u; \theta^+, \bar{\theta}^+)$ depending only on the harmonic projected odd variables $\theta^{+a} = \theta^{\alpha a} u_{\alpha}^+$, $\bar{\theta}^{+a'} = \bar{\theta}^{\alpha a'} u_{\alpha}^+$. They are gauge fields for the harmonic derivative D^{++} defined in (6.4) and for the space-time derivative $u^{+\alpha}(\partial_x)_{\alpha\dot{\alpha}}$, respectively. The linearized action is a hybrid of (3.21) and (6.7):

$$S_{d=8} = \int d^4x d^4z_{\text{an}} du d^2\theta^+ d^2\bar{\theta}^+ \left(A^{++} \partial_x^{+\dot{\alpha}} A_{\dot{\alpha}}^+ - \frac{1}{2} A^{+\dot{\alpha}} D^{++} A_{\dot{\alpha}}^+ \right) - \frac{\omega}{2} \int d^4x d^4z d^4\theta d^4\bar{\theta} du_1 du_2 \frac{A^{++}(x, z_{1\text{an}}, \theta_1^+, \bar{\theta}_1^+, u_1) A^{++}(x, z_{2\text{an}}, \theta_2^+, \bar{\theta}_2^+, u_2)}{(u_1^+ u_2^+)^2}, \quad (6.15)$$

where $\theta_i^+ = \theta^{\alpha}(u_i)_{\alpha}^+$, $\bar{\theta}_i^+ = \bar{\theta}^{\alpha}(u_i)_{\alpha}^+$ and $z_{i\text{an}} = z + \bar{\theta}_i^+ \theta_i^- + \bar{\theta}_i^- \theta_i^+$, with $i = 1, 2$. The field content of the theory is seen in the WZ gauge for A^{++} and after solving the analog $D^{++} A_{\dot{\alpha}}^+ = (\partial_x)_{\alpha\dot{\alpha}}^+ A^{++}$ of the kinematical constraint (3.47) on $A_{\dot{\alpha}}^+$ (bosons only and Abelian approximation)

$$A^{++} = (\theta^+)^2 \phi(x, z) + (\bar{\theta}^+)^2 \bar{\phi}(x, z) - 2\theta^{+a} \bar{\theta}^{+a'} \mathcal{A}_{aa'}(x, z) + 3(\theta^+)^2 (\bar{\theta}^+)^2 u^{-\alpha} u^{-\beta} G_{\alpha\beta}(x, z) \\ A_{\dot{\alpha}}^+ = u^{+\alpha} \mathcal{A}_{\alpha\dot{\alpha}}(x, z) + (\theta^+)^2 u^{-\alpha} (\partial_x)_{\alpha\dot{\alpha}} \phi + (\bar{\theta}^+)^2 u^{-\alpha} (\partial_x)_{\alpha\dot{\alpha}} \bar{\phi} - 2\theta^{+a} \bar{\theta}^{+a'} u^{-\alpha} \mathcal{F}_{aa'\alpha\dot{\alpha}} \\ + (\theta^+)^2 (\bar{\theta}^+)^2 u^{-\alpha} u^{-\beta} u^{-\gamma} \partial_{\alpha\dot{\alpha}} G_{\beta\gamma}. \quad (6.16)$$

Here $\mathcal{A}_{aa'}$, $\mathcal{A}_{\alpha\dot{\alpha}}$ are the two halves of the $d = 8$ gauge field and $\phi, \bar{\phi}$ are the two scalars of $d = 8$ SYM. The field $G_{\alpha\beta}$ is a Lagrange multiplier which becomes identified on shell with part of the $d = 8$ field strength, $\omega G_{\alpha\beta} = \mathcal{F}_{\alpha\beta} = (\partial_x)_{\alpha\dot{\alpha}}^+ \mathcal{A}_{\beta\dot{\alpha}}$. Another part, $\mathcal{F}_{aa'\alpha\dot{\alpha}} = (\partial_x)_{\alpha\dot{\alpha}} \mathcal{A}_{aa'} - (\partial_z)_{aa'} \mathcal{A}_{\alpha\dot{\alpha}}$, appears on shell as a component of $A_{\dot{\alpha}}^+$ in (6.16). The last part $\mathcal{F}_{aa'bb'} = (\partial_z)_{aa'} \mathcal{A}_{bb'} - (\partial_z)_{bb'} \mathcal{A}_{aa'}$ is contained in the second line in (6.15). The action (6.15) exhibits only half of $d = 8$ supersymmetry, the other half is realized non-linearly.

This master $d = 8$ theory gives rise to two $d = 4$ reductions. In the first we suppress the dependence on the bosonic space-time coordinates $z^{a'a}$ and combine the odd coordinates into $\theta^{\alpha A} = (\theta^{\alpha a}, \bar{\theta}^{\alpha a'})$ carrying an $SU(4)$ R-symmetry index. The gauge field $\mathcal{A}_{aa'}$ is unified with the two scalars $\phi, \bar{\phi}$ to form the sextuplet of scalars ϕ^{AB} . Thus, we recover the LHC formulation of $\mathcal{N} = 4$ SYM from this paper. In the second case, we suppress the dependence on $x^{\dot{\alpha}\alpha}$ instead. The gauge field $\mathcal{A}_{\alpha\dot{\alpha}}$ becomes the four scalars of the $\mathcal{N} = 2$ matter HM, while $\phi, \bar{\phi}$ are part of the $\mathcal{N} = 2$ gauge multiplet. In the second scenario the Lorentz index α becomes the $SU(2)$ R-symmetry index, while $\dot{\alpha}$ is identified with the Pauli-Gürsey automorphism of the HM. The gluinos are split into halves, chiral in A^{++} and anti-chiral in $A_{\dot{\alpha}}^+$ in the first scenario, or $SU(2) \times SU(2) \times U(1)$ halves in the second.

The Chern-Simons Lagrangian loses its first term in the second scenario due to the absence of x , thus becoming the HM Lagrangian. The space-time variable x in the first case is supposed to be in the chiral basis, but because of the absence of the antichiral $(\tilde{\theta}^{\dot{\alpha}a}, \tilde{\bar{\theta}}^{\dot{\alpha}a'})$ we cannot tell the difference. In the second scenario the space-time variable z is in the G-analytic basis (6.2), and we can see it because both $\theta^{\alpha a}$ and $\bar{\theta}^{\alpha a'}$ are present.

7 Conclusions

In this paper we have presented a formulation of $\mathcal{N} = 4$ SYM in LHC superspace. It is based on the use of auxiliary variables (LHs) which allow us to have the chiral half of the supersymmetry manifest, together with the full R-symmetry $SU(4)$. The other half is realized in a rather non-trivial way on the LHC dynamical semi-superfields. In the twin paper [1] we will use it to construct the full non-chiral stress-energy tensor supermultiplet in terms of LHC semi-superfields. This result, together with the Feynman rules derived in the present paper will allow us to compute the correlation functions of the full stress-tensor multiplet at Born level.

One issue that we mentioned in the beginning of the Introduction has not been addressed in this work. It is the notoriously difficult ‘ $\mathcal{N} = 4$ barrier’ for an off-shell formulation of SYM theory with manifest supersymmetry. In this paper we discussed two approaches where only half of the $\mathcal{N} = 4$ supersymmetry is manifest – the chiral half in LHC superspace or the $\mathcal{N} = 2$ half in the original harmonic superspace approach of [4]. It is also possible to have three quarters of the supersymmetry off shell in the $\mathcal{N} = 3$ harmonic superspace approach of [10], but $\mathcal{N} = 4$ has so far resisted all attempts. Could LHs or twistors help break this barrier? A natural idea seems to be to use LHs on both halves of the Lorentz group $SO(4) \sim SU(2)_L \times SU(2)_R$. This approach, known also as ‘ambitwistor’, has been advocated in [43, 44, 45], but it seems that it again stops at the step $\mathcal{N} = 3$. Hopefully, the future will tell us what else we need.

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Appendices

A Lorentz harmonics and harmonic distributions

In this Appendix we give a summary of the main formulas of the harmonic analysis on $SU(2)$. More details can be found in [30].

A.1 Harmonic coset and invariant integral

In this paper we are dealing with harmonics u_α^\pm on one half of the Euclidean Lorentz group $SO(4) \sim SU(2) \times SU(2)$. To give an explicit parametrization of the coset it is convenient to use stereographic coordinates:

$$\|u\| = \begin{pmatrix} u_1^+ & u_1^- \\ u_2^+ & u_2^- \end{pmatrix} = \frac{1}{\sqrt{1+tt}} \begin{pmatrix} e^{i\psi} & -\bar{t}e^{-i\psi} \\ te^{i\psi} & e^{-i\psi} \end{pmatrix}, \quad 0 \leq \psi < 2\pi. \quad (\text{A.1})$$

The phase ψ is the coordinate on the subgroup $U(1)$. The harmonic functions (2.6) are by definition homogeneous in $e^{i\psi}$:

$$f^{(q)}(t, \bar{t}, \psi) = e^{iq\psi} f^{(q)}(t, \bar{t}), \quad (\text{A.2})$$

thus they are equivalent to functions on the coset $SU(2)/U(1) \sim S^2$ parametrized by the complex variable t .

In the parametrization (A.1) the invariant integral on $SU(2)$ has the form

$$\int du f^{(q)}(u) \equiv \frac{i}{4\pi^2} \int_0^{2\pi} d\psi \int \frac{dt \wedge d\bar{t}}{(1+tt)^2} f^{(q)}(t, \bar{t}, \psi). \quad (\text{A.3})$$

The invariant measure is obtained by multiplying the three independent Cartan forms (up to normalization), $du = (u^{+\alpha} du_\alpha^+)(u^{-\beta} du_\beta^-)(u^{-\gamma} du_\gamma^+)$. Obviously, if $q \neq 0$ the ψ integral in (A.3) vanishes, so we derive our first integration rule:

$$\int du f^{(q)}(u) = 0 \quad \text{if } q \neq 0. \quad (\text{A.4})$$

The second rule

$$\int du 1 = 1 \quad (\text{A.5})$$

is just the normalization condition for the integral in (A.3). Finally, our third rule is

$$\int du u_{(\alpha_1}^+ \dots u_{\alpha_n}^+ u_{\beta_1}^- \dots u_{\beta_n}^- = 0 \quad \text{for } n \geq 1. \quad (\text{A.6})$$

It follows from the fact that the harmonic variables u_α^\pm transform under the fundamental representation of $SU(2)$, while the measure is $SU(2)$ invariant; thus the left-hand side in (A.6) must be an invariant, constant and totally symmetric tensor of $SU(2)$, which does not exist. The three rules are summarized in eq. (2.10).

An important property of the harmonic integral (A.3) is the vanishing of the integral of a total derivative (recall (2.11)):

$$\int du \partial^{++} f^{(-2)}(u) = 0. \quad (\text{A.7})$$

It follows from the facts that the integral projects out the singlet part of the integrand whereas the charged function $f^{(-2)}$ in (A.7) does not contain an $SU(2)$ singlet. A more

direct proof can be given using the parametrization (A.1) of the harmonic variables, the invariant measure (A.3) and the expression for ∂^{++} in this parametrization

$$\partial^{++} = e^{2i\psi} \left[-(1+t\bar{t}) \frac{\partial}{\partial \bar{t}} + \frac{it}{2} \frac{\partial}{\partial \psi} \right]. \quad (\text{A.8})$$

Substituting (A.3), (A.8) and (A.2) into (A.7), one finds

$$\begin{aligned} \int du \partial^{++} f^{(-2)}(u) &= \frac{i}{4\pi^2} \int_0^{2\pi} d\psi \int \frac{dt \wedge d\bar{t}}{(1+t\bar{t})^2} \left[-(1+t\bar{t}) \frac{\partial f}{\partial \bar{t}} + tf \right] \\ &= \frac{1}{2\pi i} \int dt \wedge d\bar{t} \frac{\partial}{\partial \bar{t}} \left(\frac{f}{1+t\bar{t}} \right) = 0. \end{aligned} \quad (\text{A.9})$$

The last integral vanishes since the function $f(t, \bar{t})$ satisfies suitable integrability condition, being globally defined on S^2 .

A.2 Harmonic distributions

The first singular harmonic distribution that we need is the delta function. It is defined by the property

$$\int dv \delta^{(q,-q)}(u, v) f^{(q)}(v) = f^{(q)}(u), \quad (\text{A.10})$$

where $f^{(q)}(u)$ is a test function. In the parametrization (A.1) the harmonic delta function has the form

$$\delta^{(q,-q)}(u_1, u_2) = \pi e^{iq(\psi_1 - \psi_2)} (1 + t_1 \bar{t}_1)^2 \delta^2(t_1 - t_2), \quad (\text{A.11})$$

where $\delta^2(t) \equiv \delta(t, \bar{t})$ is the delta function on the complex plane. The harmonic delta functions can be differentiated in a natural way, e.g.,

$$\partial_2^{++} \delta^{(q,-q)}(u_1, u_2) = -\partial_1^{++} \delta^{(q-2, 2-q)}(u_1, u_2). \quad (\text{A.12})$$

Another important distribution is

$$\frac{1}{(u_1^+ u_2^+)} = -\frac{1}{(u_2^+ u_1^+)}, \quad (\text{A.13})$$

where

$$(u_1^+ u_2^+) \equiv u_1^{+\alpha} u_{2\alpha}^+ = e^{i(\psi_1 + \psi_2)} \frac{t_2 - t_1}{\sqrt{(1+t_1 \bar{t}_1)(1+t_2 \bar{t}_2)}}. \quad (\text{A.14})$$

Acting on it, the harmonic derivative ∂^{++} produces a delta function:

$$\begin{aligned} \partial_1^{++} \frac{1}{(u_1^+ u_2^+)} &= e^{i(\psi_1 - \psi_2)} (1 + t_1 \bar{t}_1)^{3/2} (1 + t_2 \bar{t}_2)^{1/2} \frac{\partial}{\partial \bar{t}_1} \frac{1}{t_1 - t_2} \\ &= \pi e^{i(\psi_1 - \psi_2)} (1 + t_1 \bar{t}_1)^2 \delta^2(t_1 - t_2) \\ &= \delta^{(1,-1)}(u_1, u_2). \end{aligned} \quad (\text{A.15})$$

Here we assume that the singular distribution t^{-1} is defined so that the relation

$$\frac{\partial}{\partial t} \frac{1}{t} = \pi \delta^2(t) \quad (\text{A.16})$$

holds. In a similar manner, one can define the distribution $1/(u_1^+ u_2^+)^2$ with the property

$$\partial_1^{++} \frac{1}{(u_1^+ u_2^+)^2} = \partial_1^{--} \delta^{(2,-2)}(u_1, u_2). \quad (\text{A.17})$$

The chargeless harmonic delta function can be written as a complex delta function with argument $z \equiv (u_1^+ u_2^+)$:

$$\delta^{(0,0)}(u_1, u_2) = \pi \delta^2((t_1 - t_2)(1 + t_1 \bar{t}_1)^{-2}) = \pi \delta^2((u_1^+ u_2^+)) , \quad (\text{A.18})$$

using the fact that $\delta^2(t) \equiv \delta(t, \bar{t})$ is invariant under phase rotations of the complex argument. Note also the relation $\delta^{(q,-q)}(u_1, u_2) = (u_1^+ u_2^-)^q \delta^{(0,0)}(u_1, u_2)$.

A.3 Mixed harmonic/space-time distributions

In Sect. 5 we encounter the singular distribution $1/x^{\dot{--}}$, i.e. the inverse of one of the light-cone projections (5.5). The latter can be treated as pairs of complex conjugate variables,

$$z \equiv x^{\dot{++}}, \quad \bar{z} \equiv x^{\dot{--}} \quad \text{and} \quad \zeta \equiv x^{\dot{+-}}, \quad \bar{\zeta} \equiv x^{\dot{-+}}. \quad (\text{A.19})$$

Here we assume that x is a hermitian matrix while the left- and right-handed LHs u^\pm and λ^\pm have the conjugation properties (2.5) and (5.3), respectively. In terms of the new variables (A.19) the space-time derivative becomes (see (5.4))

$$\partial_{\alpha\dot{\alpha}} = \lambda_{\dot{\alpha}}^+ u_{\alpha}^+ \partial_z + \lambda_{\dot{\alpha}}^- u_{\alpha}^- \partial_{\bar{z}} + \lambda_{\dot{\alpha}}^+ u_{\alpha}^- \partial_{\bar{\zeta}} + \lambda_{\dot{\alpha}}^- u_{\alpha}^+ \partial_{\zeta},$$

or projecting with $u^{+\alpha}$,

$$\partial_{\dot{\alpha}}^+ = \lambda_{\dot{\alpha}}^+ \partial_{\bar{z}} + \lambda_{\dot{\alpha}}^+ \partial_{\bar{\zeta}}. \quad (\text{A.20})$$

From this we find

$$\partial_{\dot{\alpha}}^+ \frac{1}{x^{\dot{--}}} = \left(\lambda_{\dot{\alpha}}^+ \partial_{\bar{z}} + \lambda_{\dot{\alpha}}^+ \partial_{\bar{\zeta}} \right) \frac{1}{\bar{\zeta}} = \pi \lambda_{\dot{\alpha}}^+ \delta^2(x^{\dot{--}}). \quad (\text{A.21})$$

Here the delta function is to be understood in the complex sense, $\delta^2(x^{\dot{--}}) \equiv \delta(\zeta, \bar{\zeta})$.

In the light-cone basis the harmonic derivative ∂^{++} becomes

$$\partial^{++} = \partial_u^{++} + x^{\dot{--}} \partial_{\bar{z}} + x^{\dot{+-}} \partial_{\bar{\zeta}}, \quad (\text{A.22})$$

so that $\partial^{++} x^{\dot{\pm-}} = x^{\dot{\pm-}}$. With this we find

$$\partial^{++} \frac{1}{x^{\dot{--}}} = x^{\dot{--}} \partial_{\bar{\zeta}} \frac{1}{\bar{\zeta}} = \pi x^{\dot{--}} \delta^2(x^{\dot{--}}). \quad (\text{A.23})$$

Finally, the mixed integral $\int du \delta^2(x^{\dot{++}})$ is to be understood as follows. The support of the complex delta function $\delta^2(x^{\dot{++}}) = \delta(x^{\dot{++}}, x^{\dot{--}})$ is given by

$$x^{\dot{++}} = 0 \rightarrow x^{\dot{+}\alpha} = au^{+\alpha}, \quad x^{\dot{--}} = 0 \rightarrow x^{\dot{-}\alpha} = bu^{-\alpha}.$$

Remembering the defining properties (2.5), we find $a^* = b$ and $|a|^2 = x^2$ since $x^{\dot{+}\alpha} x_{\dot{\alpha}}^{\dot{-}} = x^2 \lambda^{\dot{+}\dot{\alpha}} \lambda_{\dot{\alpha}}^{\dot{-}} = x^2$. We can then choose $a = b = \sqrt{x^2}$ up to an insignificant $U(1)$ phase (the freedom in the definition of the $SU(2)$ harmonics). This yields the identifications

$$x^{\dot{+}\alpha} = \sqrt{x^2} u^{+\alpha}, \quad x^{\dot{-}\alpha} = \sqrt{x^2} u^{-\alpha} \quad (\text{A.24})$$

and $x^{\dot{--}} = -\sqrt{x^2}$. So, the harmonic integration produces a Jacobian factor,

$$\int du \delta^2(x^{\dot{++}}) = \frac{1}{x^2} \int du \delta^2\left(\frac{x^{\dot{+}} u^+}{\sqrt{x^2}}\right) = \frac{1}{\pi} \frac{1}{x^2} \int du \delta^{(0,0)}(v, u) = \frac{1}{\pi} \frac{1}{x^2}, \quad (\text{A.25})$$

where we have used (A.18), treating $v^+ \equiv x^{\dot{+}}/\sqrt{x^2}$ as a second harmonic variable.

B Dictionary twistors-harmonics

Here we give a short dictionary from the twistor approach (notation of ref. [23]) and the present LHC approach (see also [38] for an earlier version of such a comparison). Eq. (B2.9) (**B** for Boels et al) in [23] gives the matching rules:

$$\pi_{A'} \rightarrow u_{\alpha}^+, \quad \hat{\pi}_{A'} \rightarrow -u_{\alpha}^-, \quad \omega^A \rightarrow x^{+\dot{\alpha}}, \quad \psi^i \rightarrow \theta^{+A}$$

with $i \rightarrow A$ being the $SU(4)$ index. In addition, we have the $SU(2)$ unit determinant defining condition (2.5) which makes writing $(\pi \cdot \hat{\pi})$ in (B2.10), etc. unnecessary.

The derivatives in (B2.11) are matched by

$$\bar{\partial}_0 \rightarrow \partial^{++}, \quad \bar{\partial}_A \rightarrow \partial_{\dot{\alpha}}^+.$$

The super-connection \mathcal{A} in [23] is expanded over the basis of one-forms (B2.10), $\mathcal{A} = \bar{e}^0 \mathcal{A}_0 + \bar{e}^A \mathcal{A}_A$. The matching rule is

$$\mathcal{A}_0 \rightarrow A^{++}, \quad \mathcal{A}_A \rightarrow A_{\dot{\alpha}}^+.$$

The expansion of the superfield \mathcal{A} in (B3.1) is exactly matched by our (3.37) (and similarly for A^{++} , from which we see only the truncated version in the WZ gauge (3.36)). Note that in the harmonic approach it is important to keep track of the $U(1)$ charges \pm . Every harmonic function $f(u)$ must be homogeneous of a given degree in this charge. In the twistor approach the notion of charge is replaced by “holomorphic weight” (meaning positive charge).

Eq. (B3.8) is supposed to mean the WZ gauge (3.36) for A^{++} . The gauge (3.36) is more than obvious, assuming that all fields have harmonic expansions like (2.6). What we then see in (B3.9) is just what is written in (3.36), (3.37).

The log det term in the action (B3.20) is the exact match of our (3.22), (3.23), with $(\pi_1 \cdot \pi_2) \rightarrow (u_1^+ u_2^+)$ and $\pi \cdot d\pi \rightarrow du$. We recall that the harmonic integral $\int du f(u) = f_0$ of a function with zero charge is defined to pick the first (singlet) term in its harmonic expansion (2.6). In the twistor approach one has to choose the appropriate integration measure. As an example, the integral in (B3.21) makes use of the measure (B3.22) (“Kähler form”). The harmonic version of this is very simple:

$$\begin{aligned} & \int d^8\theta \frac{du_1 du_2}{(u_1^+ u_2^+)^2} (\theta_1^+)^4 u_1^{-\alpha} u_1^{-\beta} G_{\alpha\beta} (\theta_2^+)^4 u_2^{-\gamma} u_2^{-\delta} G_{\gamma\delta} \\ &= \int du_1 du_2 (u_1^+ u_2^+)^2 u_1^{-\alpha} u_1^{-\beta} G_{\alpha\beta} u_2^{-\gamma} u_2^{-\delta} G_{\gamma\delta} = \frac{1}{9} G_{\alpha\beta} G^{\alpha\beta}. \end{aligned} \quad (\text{B.1})$$

At the last step we simply extracted the singlet from the product $G_{\alpha\beta} G_{\gamma\delta}$, no need to go into the details of the integration. This clearly shows the advantage of preserving the manifest Lorentz symmetry in the harmonic approach.

Next, we compare the gauge-fixing conditions. Just before (B3.28) we read $\eta^A \bar{\partial}_{A\perp} \mathcal{A} = 0$ where η^A is an arbitrary but fixed commuting chiral spinor (not to be confused with the antichiral $\pi_{A'}$). We translate this condition as (5.1). The spinor $\lambda^{\dot{\alpha}}$ has no $U(1)$ charge and hence we cannot mix $\lambda^{\dot{\alpha}} A_{\dot{\alpha}}^+$ with A^{++} , unless we put in front of it a negative-charged parameter $\mu^- = \mu^{\alpha} u_{\alpha}^-$. We can thus consider the following generalized gauge:

$$\lambda^{\dot{\alpha}} A_{\dot{\alpha}}^+ + \mu^{\alpha} u_{\alpha}^- A^{++} = 0. \quad (\text{B.2})$$

The pair $Z_* = (\lambda^{\dot{\alpha}}, \mu_{\alpha})$ forms a four-component twistor. The gauge (A3.9) (A for Adamo et al) in [26] makes use of such a ‘reference twistor’, but one should be allowed to set $\mu^{\alpha} = 0$ (see the special choice $Z_* = (i^A, 0, 0)$ just above (A3.9)). Indeed, we can write $\mu^{\alpha} = \lambda_{\dot{\alpha}} x_*^{\dot{\alpha}\alpha}$ and then we should be able to set $x_* = 0$ by a translation in Minkowski space. However, it is hard for us to find an analog of the fermionic part of the so-called ‘reference supertwistor’ of the CSW gauge given in [26]. Its role is to maintain the appearance of superconformal symmetry in the gauge, but we know that the very presence of the fixed reference supertwistor Z_* breaks all symmetries. They are only restored once one has shown how the gauge-fixing parameter drops out of the sum of all Feynman graphs. This is not so simple, as discussed in [11] and [1].

Finally, we find it not very easy to understand page 14 in [26] with the derivation of the propagators in the presence of Z_* . It starts by writing the propagator equation for the truncated (gauge-fixed) CS action. This equation has just a delta function as its right-hand side, while we would expect that the gauge fixing yields additional contact terms. Then it gives the solution (A3.11). It is not immediately obvious, but nevertheless possible to extract our propagators $\langle A^{++} A^{++} \rangle$ and $\langle A_{\dot{\alpha}}^+ A^{++} \rangle$ from it. The tricky point is to convert from twistors back to space-time variables.¹⁶ Last but not least, the equation after

¹⁶We thank Lionel Mason for the explanation.

(A3.11) states that this propagator satisfies not quite the required equation, but produces “essentially vanishing error terms”. It appears in fact that the Green’s function equation at the top of page 14 is not correct, but the solution (A3.11) is. The authors of [26] remark that choosing $Z_* = (0, \lambda, 0)$ would lead to ‘error terms’ of the type $\delta(\pi_\alpha)\delta(\mu_{\dot{\alpha}} + \lambda_{\dot{\alpha}}) = 0$ since the twistor $\pi_\alpha \neq 0$ by definition. This confirms once more that the most natural gauge-fixing parameter is $\lambda_{\dot{\alpha}}$.

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